Energy scaling laws for conically constrained thin elastic sheets

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Abstract

We investigate low-energy deformations of a thin elastic sheet subject to a displacement boundary condition consistent with a conical deformation. Under the assumption that the displacement near the sheet's center is of order $h|\log h|$, where $h \ll 1$ is the thickness of the sheet, we establish matching upper and lower bounds of order $h^2|\log h|$ for the minimum elastic energy per unit thickness, with a prefactor determined by the geometry of the associated conical deformation. These results are established first for a 2D model problem and then extended to 3D elasticity.

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1 Introduction

1.1 Motivation, contribution, and remaining questions

In this paper, we investigate the following question:

- What is the limiting behavior of a thin elastic sheet subject to a displacement boundary condition consistent with a conical deformation? In particular:
  - What is the elastic energy scaling law for such a sheet?
  - Do deformations satisfying this scaling law converge in some sense to the associated conical deformation?

We provide partial answers to these questions, demonstrating that:

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- Under the additional assumption that the displacement near the sheet's center is at most $Ch|\log h|$, where $h \ll 1$ is the thickness of the sheet, the minimum elastic energy per unit thickness satisfies matching upper and lower bounds of order $h^2|\log h|$, with a prefactor determined by the geometry of the associated conical deformation.

With a stronger hypothesis on the displacement of the sheet's center, Müller and Olbermann have improved our result by giving an estimate for the leading order correction to our bounds [8]. (See Section 1.3 for further discussion of our results and connections with [8].)

It is natural to conjecture that an $h^2|\log h|$ energy scaling law holds even without a restriction on the deformation at the sheet's center. Such a result is, however, beyond the scope of our methods (except as indicated in Remark 2 following the proof of Theorem 1 in Section 2).

In the real world, conical deformations can arise without fixing displacement boundary conditions. For example, a conical deformation known as the d-cone forms when a thin elastic sheet is placed on top of an open cylinder and a downward force is applied at the center of the sheet [3]. Our work can be seen as a mathematical idealization of the d-cone experiment. The physics literature includes numerous studies of nearly conical deformations subject to geometric boundary conditions. Important contributions include those of Pomeau and Ben Amar [3] and Cerda and Mahadevan [4]. In [3], it is shown that the d-cone arises as the surface which minimizes bending energy among those surfaces satisfying a conical boundary displacement condition and which are developable outside of an inner region. In [4], the d-cone is modeled as an inextensible surface subject to the constraint that its edge lie above an open cylinder and the resulting shape is found by solving a free boundary problem for the edge deformation. For a more complete survey of the literature and many more references, we refer to the excellent review by Witten [10]. It should perhaps be emphasized that the results of the present paper (the energy scaling law, and the approximately conical character of the deformation) are assumed rather than proved in the physics literature.

1.2 Two and three-dimensional elastic energies

In Section 2, we establish our results for a 2D model energy. This choice of energy simplifies our analysis while capturing the essential features of our argument. We then extend our results to more general 3D elastic energies in Section 3. In the present section, we describe our 2D and 3D elastic energies and provide intuition as to why low-energy deformations satisfying conical boundary conditions are nearly conical away from the sheet's center.

Given a thin elastic sheet

$$\Omega_h = \Omega \times (-\frac{h}{2}, \frac{h}{2}) \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ and $h \ll 1$, our 2D model energy

$$\int_\Omega |\nabla u^T \nabla u - I_2|^2 + h^2 |\nabla\nabla u|^2 \, dx, \tag{1.2}$$

arises as an upper bound for the asymptotic behavior of the model energy

$$\frac{1}{h} \int_{\Omega \times (-\frac{h}{2}, \frac{h}{2})} |(\nabla \phi^T \nabla \phi) - I_3|^2 \, dx \tag{1.3}$$

for maps satisfying the Kirchhoff-Love ansatz

$$\phi(x,y,z) = u(x,y) + zN(x,y) \text{ where } N(x,y) = \frac{u_x \times u_y}{|u_x \times u_y|}. \tag{1.4}$$

Here, I_2 is the 2×2 identity matrix, I_3 is the 3×3 identity matrix, ∇u is the 3×2 Jacobian $\left(\frac{\partial u^i}{\partial x_j}\right)$, $1 \le i \le 3, 1 \le j \le 2$, and $\nabla\phi$ is the 3×3 Jacobian $\left(\frac{\partial \phi^i}{\partial x_j}\right)$, $1 \le i,j \le 3$. To see how (1.2) arises as an upper bound for (1.3), first integrate in the z variable and drop higher order terms. This leads to a functional of the form

$$\int_\Omega |\nabla u^T \nabla u - I_2|^2 + ch^2\left(|u_{xx}\cdot N|^2 + |u_{xy}\cdot N|^2 + |u_{yy}\cdot N|^2\right)\,dx$$

where c is a numerical constant. Simplifying the above expression by replacing the bending energy $|u_{xx}\cdot N|^2 + |u_{xy}\cdot N|^2 + |u_{yy}\cdot N|^2$ with $|\nabla\nabla u|^2$ and replacing ch^2 by h^2 leads to the model (1.2). We are not the first to use the two-dimensional model (1.2) as a laboratory for understanding the behavior of thin sheets; see for example [2, 5].

In Section 3, we justify the simplifications made in deriving the energy (1.2) by extending our results to 3D elastic energies

$$\int_{\Omega_h} W(\nabla u_h)\,dx.$$

Since we make use of results from [6], we assume that the 3D elastic energy $W : M^{3\times 3} \to \mathbb{R}$ satisfies the conditions imposed there:

1. $W \in C^0(M^{3\times 3}), W \in C^2$ in a neighborhood of SO(3),
2. W is frame indifferent: $W(F) = W(RF)$ for all $F \in M^{3\times 3}$ and all $R \in \mathrm{SO}(3)$.
3. $W(F) \ge C\mathrm{dist}^2(F, \mathrm{SO}(3)),\ W(F) = 0$ if $F \in \mathrm{SO}(3)$.

Study of the energy (1.2) yields insight as to why low-energy deformations subject to a conical boundary condition are in fact approximately conical. The energy (1.2) consists of two terms, the non-convex membrane energy $|\nabla u^T\nabla u - I_2|^2$ and the bending energy $h^2|\nabla\nabla u|^2$. The membrane energy term indicates the preference of the midplane to deform isometrically, while the bending energy term penalizes variation in the normal vector field to the surface $u(\Omega)$ and accounts for the stretching of cross sections of the sheet which are parallel to the midplane.

Since only rigid motions achieve zero energy, the minimization of (1.2), subject to boundary conditions, typically involves a trade-off between the bending and stretching contributions. In order to understand this trade-off, observe that for $h \ll 1$ the bending term functions as a singular perturbation, indicating the sheet's preference to bend rather than stretch. This suggests that low energy deformations satisfy

$$|\nabla u^T\nabla u - I_2|^2 \approx 0 \tag{1.5}$$

in all of Ω, except possibly in a small region in which ∇u undergoes rapid change. A conical deformation smoothed near its tip is an example of a deformation satisfying (1.5).

1.3 Statement of results

For the remainder of the paper, we assume that the reference configuration of our sheet is given by

$$B_{1,h} = B_1 \times (-\frac{h}{2}, \frac{h}{2}) \tag{1.6}$$

where

$$B_r = \{x \in \mathbb{R}^2 : |x| < r\}. \tag{1.7}$$

Setting

$$E_h(u) = \int_{B_1} |\nabla u^T \nabla u - I_2|^2 + h^2 |\nabla\nabla u|^2 \, dx, \tag{1.8}$$

in Section 2 we prove the scaling law

Theorem 1. Let $g \in C^2(\partial B_1)$, $g : \partial B_1 \to S^2$ be a unit-speed curve, and suppose $P \in \mathbb{R}^3$ satisfies

$$|P| \leq Ch|\log_2 h|^\alpha \tag{1.9}$$

for some $0 \leq \alpha < 1/2$ and $C > 0$. Then

$$\lim_{h\to 0} \frac{1}{h^2|\log_2 h|} \min_{\substack{u \in W^{2,2}(B_1);\, u=g \text{ on } \partial B_1 \\ u(0)=P}} E_h(u) = E.$$

The constant E is given by

$$E = \int_{B_1 \setminus B_{1/2}} |\nabla^2 V|^2 \, dx,$$

where $V(x) = |x| g(x/|x|)$.

Remark 1. *The requirement that $g : \partial B_1 \to S^2$ be a unit-speed curve is motivated by our expectation that low energy deformations satisfy (1.5). Theorem 1 will be proved in Section 2. In Section 2, will also establish that $u_h \to |x|g(x/|x|)$ in $H^1(B_1)$ as h goes to 0, whenever u_h satisfies $E_h(u_h) \leq Ch^2 \log(h)$ for some C (see Proposition 1).*

Theorem 1 is established by proving the lower bound

$$E \leq \liminf_{h\to 0} \frac{1}{h^2|\log_2 h|} \min_{\substack{u \in W^{2,2}(B_1);\, u=g \text{ on } \partial B_1 \\ u(0)=P}} E_h(u)$$

and the upper bound

$$\limsup_{h\to 0} \frac{1}{h^2|\log_2 h|} \min_{\substack{u \in W^{2,2}(B_1);\, u=g \text{ on } \partial B_1 \\ u(0)=P}} E_h(u) \leq E.$$

The upper bound is achieved by a smooth deformation which agrees with the conical map $|x|g\left(\frac{x}{|x|}\right)$ except within the ball of radius $h|\log(h)|^\alpha$ centered at the origin. In order to prove the lower bound, we use the membrane energy to control the stretching of line segments. Our starting point is the observation that a low energy deformation u_h must satisfy

$$\left|\frac{\partial u_h}{\partial r}\right| \approx 1, \tag{1.10}$$

except possibly on a small set. Due to the boundary conditions and (1.9), it follows that a deformation satisfying (1.10) closely approximates the conical map $|x|g\left(\frac{x}{|x|}\right)$. We complete the proof by showing that any map with this property must have bending energy at least of the same order, $h^2|\log(h)|$, as that of the trial function used in the proof of the upper bound.

In formulating Theorem 1, we have chosen to focus on a somewhat special problem: u is defined on the unit disk, with a constraint on its value at 0. This choice is convenient, but probably not necessary. Arguments similar to ours probably could be applied in a less symmetric setting, e.g. when u is constrained at some point $x_0 \neq 0$, provided the constraint and boundary conditions are consistent with a conical configuration whose apex is at $u(x_0)$.

An earlier draft of this paper contained upper and lower bounds whose prefactors did not match. We would like to thank Heiner Olbermann for suggesting the modification to our original argument which led to the improved results reported here. In recent work, Müller and Olbermann have improved Theorem 1 by estimating the leading order correction to our bounds [8].

In Section 3, we extend our results to three dimensional elasticity. Our basic strategy is the same as in Section 2, but we rely on the compactness and lower semi-continuity results from [6]. Setting

$$E_h(u_h) = \int_{B_{1,h}} W(\nabla u_h)\, dx,$$

we prove the following result.

Theorem 2. Let $g : \partial B_1 \to S^2$ be a unit speed curve, set $\tilde{g}(\theta, z) = g(\theta)$, and define the surface $s : B_1 \to \mathbb{R}^3$ in polar coordinates by $s(r,\theta) = rg(\theta)$. We have that

$$\lim_{h\to 0} \frac{1}{h^3|\log_2 h|} \min_{\substack{u\in W^{1,2}(B_{1,h})\cap C(\bar{B}_{1,h});\, \max_{\partial B_1\times(-h/2,h/2))}|u-\tilde{g}|\le Ch|\log_2 h| \\ \max_{B_{h,h}}|u|\le Ch|\log_2 h|}} E_h(u) = E$$

where $B_{h,h} = B_h \times (-\frac{h}{2}, \frac{h}{2})$, and the constant E is given by

$$E = \int_{B_1\setminus B_{1/2}} Q_2(II)\, dx'.$$

Here, Q_2 is a quadratic form on $M^{2\times 2}$, given in [6], and II is the second fundamental form of the surface s.

2 Two dimensional result

In this section, we state and establish results related to the 2D model energy (1.2). We begin with the energy scaling law, which we repeat for the reader's convenience:

Theorem 1. *Let $g \in C^2(\partial B_1)$, $g : \partial B_1 \to S^2$ be a unit-speed curve, and suppose $P \in \mathbb{R}^3$ satisfies*

$$|P| \le Ch|\log_2 h|^\alpha \tag{2.1}$$

for some $0 \le \alpha < 1/2$ and $C > 0$. Then

$$\lim_{h\to 0} \frac{1}{h^2|\log_2 h|} \min_{\substack{u\in W^{2,2}(B_1);\, u=g \text{ on } \partial B_1 \\ u(0)=P}} E_h(u) = E.$$

The constant E is given by

$$E = \int_{B_1\setminus B_{1/2}} |\nabla^2 V|^2\, dx,$$

where $V(x) = |x|g(x/|x|)$.

Hereafter, C denotes a positive constant independent of h and g.

Proof. In what follows, we suppose that h is small and set $h_* = h|\log_2 h|^\alpha$.

Step 1: Proof of the upper bound.

Since $g \in C^2(\partial B)$ and $|P| \le Ch_*$, we can find $u \in C^2(\bar{B})$ satisfying

$$\begin{cases} u(x) = |x|g(x/|x|) & \text{for } x \in B_1 \setminus B_{h_*} \\ u(0) = P & \\ |\nabla u| \le C & \text{on } B_{h_*} \\ |\nabla^2 u| \le C/h_* & \text{on } B_{h_*}. \end{cases} \tag{2.2}$$

Due to the assumptions on g, we have

$$\begin{aligned} E_h(u) &= \int_{B_{h_*}} |\nabla u^T \nabla u - I_2|^2 + h^2 \int_{B_{2h_*}} |\nabla^2 u|^2 + h^2 \int_{B\setminus B_{2h_*}} |\nabla^2 u|^2 \\ &\le Ch_*^2 + Ch^2 + h^2 \sum_{n\ge 0;\ 2^{-n-1}\ge h_*} \int_{B_{2^{-n}}\setminus B_{2^{-n-1}}} |\nabla^2 u|^2. \end{aligned} \tag{2.3}$$

On the other hand, define $V_n(x) = 2^n u(2^{-n}x)$ for $x \in B_1 \setminus B_{1/2}$. Then $V_n = V$ if $2^{-n-1} \ge h_*$ and by a change of variables, we have

$$\int_{B_{2^{-n}}\setminus B_{2^{-n-1}}} |\nabla^2 u|^2 = \int_{B_1\setminus B_{1/2}} |\nabla^2 V_n|^2 = E. \tag{2.4}$$

Combining (2.3) and (2.4) yields

$$\limsup_{h\to 0} \frac{1}{h^2|\log_2 h|} \min_{\substack{u\in W^{2,2}(B_1);\, u=g \text{ on } \partial B_1 \\ u(0)=P}} E_h(u) \le \limsup_{h\to 0} \frac{1}{|\log_2 h|} \sum_{n\ge 0;\ 2^{-n-1}\ge h_*} E = E.$$

Step 2: Proof of the lower bound.

Let $\{u_h\}$ be a sequence of deformations satisfying

$$E_h(u_h) \le Ch^2|\log(h)|. \tag{2.5}$$

We begin by using (2.5) to control the behavior of the $\{u_h\}$ near the origin. It follows from (2.5) that

$$\int_{B_1} |\nabla u_h^T \nabla u_h - I_2|^2 \le Ch^2|\log_2 h|,$$

$$\int_{B_1} |\nabla u_h|^2 \le C,$$

and

$$\int_{B_1} |\nabla^2 u_h|^2 \le C|\log_2 h|.$$

An application of the Sobolev embedding theorem [1] yields

$$\|u_h\|_{C^{0,\gamma}} \le C(\gamma)|\log_2 h|^{1/2} \tag{2.6}$$

for $0 < \gamma < 1$. This implies

$$\sup_{|x|=h} |u_h(x)| \le \sup_{|x|=h} |u_h(x) - u_h(0)| + |u_h(0)| \le C(\gamma)|\log_2 h|^{1/2}h^\gamma + C(\gamma)h|\log_2 h|^\alpha. \tag{2.7}$$

Using (2.7) and the sublinear growth of the logarithm, we see that for each $\gamma \in (0,1)$ there exists $C'(\gamma)$ such that

$$\sup_{|x|=h} |u_h(x)| \le C'(\gamma)h^\gamma. \tag{2.8}$$

Next, we demonstrate that the $\{u_h\}$ approximate the conical map $|x|g\left(\frac{x}{|x|}\right)$. Our idea is to use the membrane energy, the boundary conditions, and (2.8) to control the stretching of radial line segments from the origin. To begin, we re-write the membrane energy in polar coordinates and recall (2.5) to arrive at

$$\int_{\partial B_1}\int_0^1 |\nabla u_h^T \nabla u_h - I_2|^2 r\,dr\,d\theta = \int_{B_1} |\nabla u_h^T \nabla u_h - I_2|^2 \; dx \le Ch^2|\log_2 h|.$$

It follows that

$$\int_{\partial B_1}\int_h^1 \left(\left|\frac{\partial u_h}{\partial r}(r\theta)\right|^2 - 1\right)^2 r\,dr\,d\theta \le \int_{\partial B_1}\int_0^1 |\nabla u_h^T \nabla u_h - I_2|^2 r\,dr\,d\theta \le Ch^2|\log_2 h|. \tag{2.9}$$

Next, we separate the radial line segments from the origin into two classes, those with "small" radial stretching energy and those with "large" radial stretching energy. To do this, we set

$$A = \left\{\theta \in \partial B_1;\; \int_h^1 \left(\left|\frac{\partial u_h}{\partial r}(r\theta)\right|^2 - 1\right)^2 r\; dr \le h^2|\log_2 h|^2\right\}. \tag{2.10}$$

On those radial lines with small stretching energy, $\{r\theta : h \le r \le 1, \theta \in A\}$, we prove, due to the boundary conditions and (2.8), that the $\{u_h\}$ will be in close agreement with $|x|g\left(\frac{x}{|x|}\right)$. We will establish that

$$|u_h(r\theta) - rg(\theta)| \le C''(\gamma)\max(r^{1/2}h^{\gamma/2}, h^{\gamma}) \quad \text{for } \theta \in A,\ h \le r \le 1, \text{ and } 0 < \gamma < 1, \tag{2.11}$$

where $C''(\gamma)$ is a constant depending on γ. Since by Chebyshev's inequality and (2.9) we also have

$$|\mathcal{H}^1(A) - \mathcal{H}^1(\partial B_1)| \le C/|\log_2 h|, \tag{2.12}$$

we see from (2.11) and (2.12) that the $\{u_h\}$ approximate the conical map $|x|g\left(\frac{x}{|x|}\right)$.

In order to establish (2.11), we first prove the following lemma.

Lemma 1. *Let $v \in \mathbb{R}^3$ be such that $|v| = 1$ and $h \le r \le 1$. Suppose $f : [h, 1] \to \mathbb{R}^3$ satisfies $f(1) = v$ and $|f(h)| \le C_1 h^{\kappa}$ for some $0 < \kappa \le 1$. Then*

$$|f(r) - rv|^2 \le 2r\left(\int_h^1 \left|\left|\frac{df}{ds}\right|^2 - 1\right| ds + C_2 h^{\kappa}\right) + C_3 h^{2\kappa}$$

for some constants C_2, C_3 depending on C_1.

Proof. Due to our assumptions on f, application of the fundamental theorem of calculus and the Cauchy-Schwartz inequality lead to

$$|f(r) - rv|^2 \le 2\left(\int_h^r \left|\frac{df}{ds} - v\right| ds\right)^2 + C_3 h^{2\kappa} \le 2r \int_h^1 \left|\frac{df}{ds} - v\right|^2 ds + C_3 h^{2\kappa}.$$

Expanding the square,

$$\left|\frac{df}{ds} - v\right|^2 = \left|\frac{df}{ds}\right|^2 + 1 - 2\frac{df}{ds}\cdot v = \left|\frac{df}{ds}\right|^2 - 1 + 2\left(v - \frac{df}{ds}\right)\cdot v.$$

The boundary conditions of f then imply that

$$\left|\int_h^1 \left(v - \frac{df}{ds}\right)\cdot v\ dr\right| \le C_2 h^{\kappa}$$

and the lemma follows. □

We now establish (2.11) using Lemma 1. According to Lemma 1 and (2.8),

$$|u_h(r\theta) - rg(\theta)|^2 \le 2r\left(\int_h^1 \left|\left|\frac{\partial u_h}{\partial r}(s\theta)\right|^2 - 1\right| ds + C_2 h^{\gamma}\right) + C_3 h^{2\gamma} \quad \text{for } h \le r \le 1. \tag{2.13}$$

An application of Cauchy-Schwartz shows that

$$\int_h^1 \left|\left|\frac{\partial u_h}{\partial r}(s\theta)\right|^2 - 1\right| ds \le \left(\int_h^1 s\left|\left|\frac{\partial u_h}{\partial r}(s\theta)\right|^2 - 1\right|^2 ds\right)^{1/2}\left(\int_h^1 1/s\ ds\right)^{1/2}. \tag{2.14}$$

Using (2.14) and (2.10), we can bound the term in parentheses on the right-hand side of (2.13). Comparing this bound to the $C_3 h^{2\gamma}$ term on the right-hand side of (2.13) yields (2.11).

Using (2.11) and (2.12), we will now establish that

$$\liminf_{h\to 0} \frac{1}{|\log_2 h|}\|\nabla^2 u_h\|^2_{L^2(h\le|x|\le 1)} \ge E, \tag{2.15}$$

which will conclude the proof of Theorem 1. In order to establish (2.15), fix $\varepsilon > 0$ and $0 < \sigma < \gamma < 1$. We claim that

$$\|\nabla^2 u_h\|^2_{L^2(2^{-n-1}\le|x|\le 2^{-n})} \ge E - \varepsilon \text{ if } 2^{-n-1} \ge h^\sigma \text{ and } h \ll 1. \tag{2.16}$$

Indeed, define $V_n(x) = 2^n u_h(2^{-n}x)$. It follows from (2.11) and (2.12) that

$$\left|\left\{x \in B_1 \setminus B_{1/2}; \left|V_n(x) - |x| g(x/|x|)\right| \ge C''(\gamma) h^{(\gamma-\sigma)/2}\right\}\right| \le C/|\log_2 h|.$$

By Lemma 2 (stated and proved just below), this implies

$$\|\nabla^2 u_h\|^2_{L^2(2^{-n-1}\le|x|\le 2^{-n})} = \|\nabla^2 V_n\|^2_{L^2(B_1\setminus B_{1/2})} \ge E - \varepsilon.$$

Using (2.16) to bound from below the contributions to the bending energy from the annuli $\{x : 2^{-n-1} \le |x| \le 2^{-n}\}$, we find that

$$\liminf_{h\to 0} \frac{1}{|\log_2 h|}\|\nabla^2 u_h\|^2_{L^2(h\le|x|\le 1)} \ge \lim_{h\to 0} \frac{1}{|\log_2 h|} \sum_{n\ge 0; 2^{-n-1}\ge h^\sigma} (E-\varepsilon) = (E-\varepsilon)\sigma. \tag{2.17}$$

Since $\sigma < 1$ and $\varepsilon > 0$ are arbitrary, the conclusion follows. □

The following lemma was used in the proof of Theorem 1. Geometrically, it says that if a surface is sufficiently close to a smooth fixed surface, then the amount by which it bends must be at least comparable with that of the fixed surface.

Lemma 2. *Let Ω be a smooth bounded domain, $h > 0$, $\alpha, \beta : (0,1) \to \mathbb{R}_+$, and $v, v_h \in W^{2,2}(\Omega)$ be such that*

$$\lim_{t\to 0} \alpha(t) = \lim_{t\to 0} \beta(t) = 0$$

and

$$\left|S_h := \left\{x \in \Omega; \left|v_h(x) - v(x)\right| \ge \alpha(h)\right\}\right| \le \beta(h). \tag{2.18}$$

Then for any $\varepsilon > 0$ there exists $\delta > 0$, depending only on α, β, Ω, and v such that

$$\|\nabla^2 v_h\|_{L^2(\Omega)} \ge \|\nabla^2 v\|_{L^2(\Omega)} - \varepsilon \quad \forall\, h < \delta.$$

Proof. We prove the lemma by contradiction. Suppose that there exists $\varepsilon_0 > 0$ and a sequence $(v_{h_n}) \subset W^{2,2}(\Omega)$ satisfying (2.18) but

$$\|\nabla^2 v_{h_n}\|_{L^2(\Omega)} \le \|\nabla^2 v\|_{L^2(\Omega)} - \varepsilon_0. \tag{2.19}$$

The key step is establishing that

$$\sup_n \|v_{h_n}\|_{H^2(\Omega)} < \infty. \tag{2.20}$$

Once (2.20) is established, it follows that $v_{h_n} \rightharpoonup v$ in $W^{2,2}(\Omega)$, which results in

$$\liminf_{n\to\infty} \|\nabla^2 v_{h_n}\|_{L^2(\Omega)} \geq \|\nabla^2 v\|_{L^2(\Omega)},$$

a contradiction.

In order to establish (2.20), consider

$$u_{h_n} = v_{h_n} - (a_{h_n}^T x + b_{h_n})$$

where

$$a_{h_n} = \frac{1}{|\Omega|}\int_\Omega \nabla v_{h_n}\, dx \text{ and } b_{h_n} = \frac{1}{|\Omega|}\int_\Omega v_{h_n} - a_{h_n} x\, dx.$$

It is clear that

$$\int_\Omega \nabla u_{h,n} = 0 \quad \text{ and } \quad \int_\Omega u_{h,n} = 0.$$

An application of Poincaré's inequality, along with (2.19), then yields

$$\|u_{h_n}\|_{H^2(\Omega)} \leq C\|\nabla^2 u_{h_n}\|_{L^2(\Omega)} \leq C\|\nabla^2 v\|_{L^2(\Omega)} - C\varepsilon_0. \tag{2.21}$$

It follows that

$$u_{h_n} = v_{h_n} - (a_{h_n}^T x + b_{h_n}) \rightharpoonup u \text{ in } H^2(\Omega).$$

For $x \notin S_{h_n}$ we then have

$$|v(x) - (a_{h_n}^T x + b_{h_n}) - u(x)| < \alpha(h_n)$$

from which it follows that the $\{a_{h_n}\}$ and $\{b_{h_n}\}$ are bounded. Boundedness of $\{a_{h_n}\}$ and $\{b_{h_n}\}$, along with (2.21), leads to (2.20). □

Remark 2. *The essence of our argument for Theorem 1 is that if $u(0)$ is near 0 and $u = g$ on ∂B_1 then the image of each ray from the origin must be almost straight, because anything else costs too much membrane energy. Can one dispense with the hypothesis that $u(0)$ is near 0? Well, if the boundary curve $g(\partial B_1)$ met both $\{x \cdot e > 0\}$ and $\{x \cdot e < 0\}$ for every unit vector $e \in \mathbb{R}^3$, then the smallness of the membrane energy could be used to* prove *that $u(0)$ had to be near 0. Alas, there is no such g: a curve on S^2 with arclength 2π must lie in a halfspace*[1] *(see e.g. Lemma 19 in Chapter 6 of [9]). However, the argument just sketched can be applied in the context of the "e-cones" considered in [7].*

The next result demonstrates that low-energy deformations, subject to the boundary conditions of Theorem 1, converge in a non-oscillatory manner (strongly in $H^1(B_1)$) to the conical map $|x|g\left(\frac{x}{|x|}\right)$.

[1] We thank Heiner Olbermann for pointing this out.

Proposition 1. *Let $g : \partial B_1 \to S^2$ be a unit speed curve and suppose that a sequence of deformations $\{u_h\}$ satisfies*

$$u_h|_{\partial B_1} = g, \;\; u_h(0) \leq Ch|\log_2(h)|^\alpha \text{ for some } 0 \leq \alpha < 1/2, \text{ and } E_h(u_h) \leq Ch^2|\log_2(h)|.$$

Then $\{u_h\}$ converges strongly in $H^1(B_1)$ to the conical deformation $|x|g\left(\frac{x}{|x|}\right)$.

Proof. Let $d_h(x)$ be the test function defined by (2.2). We will prove that

$$\|u_h - d_h\|_{H^1(B_1)} \leq Ch^\alpha \text{ for } 0 \leq \alpha < 1/4.$$

We first establish the following lemma.

Lemma 3. *Let $\{u_h\}$ be as in Proposition 1. Then*

$$\|u_h - d_h\|_{L^2(B_1)} \leq Ch^\beta \text{ for } 0 \leq \beta < 1/2. \tag{2.22}$$

Proof. The bound

$$\|u_h - d_h\|_{L^2(B_{h*})} \leq Ch^\lambda \text{ for } 0 \leq \lambda < 1$$

follows from (2.6). The bound

$$\|u_h - d_h\|_{L^2(B_1 \setminus B_{h*})} \leq Ch^\beta \text{ for } 0 \leq \beta < 1/2$$

follows from integrating (2.13) in r and θ, applying Cauchy-Schwartz as in (2.14), and using the hypothesis on $E_h(u_h)$. □

Next, we establish

$$\|\nabla(u_h - d_h)\|^2_{L^2(B_1)} \leq Ch^\beta|\log_2(h)|^{1/2} \text{ for } 0 \leq \beta < 1/2. \tag{2.23}$$

Our proof of (2.23) relies on the interpolation inequality

$$\|\nabla f\|^2_{L^2(B_1)} \leq C\|f\|_{L^2(B_1)}\|\nabla\nabla f\|_{L^2(B_1)}, \tag{2.24}$$

valid for $f \in W^{2,2}(B_1)$ satisfying $f|_{\partial B_1} = 0$, applied to $f = u_h - d_h$. The estimate (2.23) follows from (2.24), (2.22) and

$$\|\nabla\nabla(u_h - d_h)\|_{L^2(B_1)} \leq C|\log_2(h)|^{1/2}. \tag{2.25}$$

In order to establish (2.25), we note that

$$\|\nabla\nabla u_h\|_{L^2(B_1)} \leq C|\log_2(h)|^{1/2},$$

due to our hypothesis on $E_h(u_h)$, and

$$\|\nabla\nabla d_h\|_{L^2(B_1)} \leq C|\log_2(h)|^{1/2}$$

by construction.

The conclusion of Proposition 1 now follows from (2.23), (2.22), and the fact that $d_h \to |x|g(x/|x|)$ in $H^1(B_1)$ as h goes to 0. □

3 Three dimensional result

In this section, we extend our two dimensional results to three dimensional deformations $u_h : B_{1,h} \to \mathbb{R}^3$ and elastic energies

$$E_h(u_h) = \int_{B_{1,h}} W(\nabla u_h)\ dx$$

where W satisfies the conditions described in Section 1.2.

Throughout this section we use the following rescalings of $u_h|_{\{2^{-j-1}<r<2^{-j},|z|<h/2\}}$: we define $u_{h,j} = 2^j u_h(x/2^j)$, $\tilde{h}_j = 2^j h$, and $v_{h,j} = u_{h,j}(x_1, x_2, \tilde{h}_j x_3)$. Performing a change of variables, we have that

$$\frac{1}{h^3}\int_{2^{-j-1}<r<2^{-j},\ |z|<h/2} W(Du_h)\ dx = \frac{1}{\tilde{h}_j^3}\int_{1/2<r<1,\ |z|<\tilde{h}_j/2} W(Du_{h,j})\ dx.$$

Using the notation $\nabla' y = y_{,1}\otimes e_1 + y_{,2}\otimes e_2$ to denote the in-plane gradient, we can perform an additional change of variables in x_3 to arrive at

$$\frac{1}{h^3}\int_{2^{-j-1}<r<2^{-j},\ |z|<h/2} W(Du_h)\ dx = \frac{1}{\tilde{h}_j^2}\int_{1/2<r<1,\ |z|<1/2} W\left(\nabla' v_{h,j}, \frac{1}{\tilde{h}_j} v_{h,j,3}\right)\ dx. \qquad (3.1)$$

The goal of this section is to prove Theorem 2, which we repeat for the reader's convenience:

Theorem 2. *Let $g : \partial B_1 \to S^2$ be a unit speed curve, set $\tilde{g}(\theta, z) = g(\theta)$, and define the surface $s : B_1 \to \mathbb{R}^3$ in polar coordinates by $s(r,\theta) = rg(\theta)$. We have that*

$$\lim_{h\to 0}\frac{1}{h^3|\log_2 h|}\min_{\substack{u\in W^{1,2}(B_{1,h})\cap C(\bar{B}_{1,h});\ \max_{\partial B_1\times(-h/2,h/2))}|u-\tilde{g}|\le Ch|\log_2 h|\\ \max_{B_{h,h}}|u|\le Ch|\log_2 h|}} E_h(u) = E$$

where $B_{h,h} = B_h \times (-\frac{h}{2}, \frac{h}{2})$, and the constant E is given by

$$E = \int_{B_1\setminus B_{1/2}} Q_2(II)\ dx'.$$

Here, Q_2 is a quadratic form on $M^{2\times 2}$, given in [6], and II is the second fundamental form of the surface s.

Proof of Theorem 2.

Step 1: Proof of the upper bound.

Let $N = s_x \times s_y$ be the unit normal to $s(x,y)$, which is well-defined for $r > 0$. According to the proof of Theorem 6.2 in [6], there exists $y_h : \{B_1 \setminus B_{1/2} \times (-1/2, 1/2)\} \to \mathbb{R}^3$ satisfying

$$y_h|_{r=1/2,1} = rg + hzN \text{ and } \lim_{h\to 0}\frac{1}{h^2}\int_{1/2<r<1,|z|<1/2} W\left(\nabla' y_h, \frac{1}{h} y_{h,3}\right)\ dx = E. \qquad (3.2)$$

In order to define a low energy sequence $u_h : B_{1,h} \to \mathbb{R}^3$, it suffices to define the corresponding $v_{h,j}$. For j satisfying $2^{-j} \geq h|\log_2(h)|^{1/4}$, set $v_{h,j} = y_{\tilde{h}_j}$. Given $\epsilon > 0$, it follows from (3.2) and (3.1) that for all such j and sufficiently small h,

$$\frac{1}{h^3}\int_{2^{-j-1}<r<2^{-j},|z|<h/2} W\left(Du_h\right)\, dx < E+\epsilon.$$

Defining $u_h|_{\{0<r<h|\log_2(h)|^{1/4},|z|<h/2\}} = rg(\theta) + \frac{r}{h|\log_2(h)|^{1/4}}zN$, we have

$$\frac{1}{h^3}\int_{r<h|\log_2(h)|^{1/4},|z|<h/2} W(Du_h)\, dx < C|\ln h|^{1/2}.$$

Putting all of this together, we have that

$$\lim_{h\to 0}\frac{1}{h^3|\log_2(h)|}\int_{B_1^h} W(Du_h) \leq \lim_{h\to 0}\frac{1}{h^3|\log_2 h|}\sum_{j\geq 0; 2^{-j}\geq h|\log_2 h|^{1/4}} (E+\varepsilon)h^3 = (E+\varepsilon).$$

Since $\epsilon > 0$ was arbitrary, we are finished.

Step 2: Proof of the lower bound.

Let u_h be a minimizer. This implies, as in the proof of Theorem 1,

$$\max_{(\theta,z)\in A,\ h\leq r\leq 1} |u_h(r\theta,z) - rg(\theta)| \leq C(\gamma)\max(r^{1/2}h^{\gamma/2}, h^\gamma) \text{ for } 0<\gamma<1, \tag{3.3}$$

and

$$|\mathcal{H}^2(A) - \mathcal{H}^2(\partial B_1\times(-h/2,h/2))| \leq Ch/|\log_2 h|, \tag{3.4}$$

where

$$A = \left\{(\theta,z)\in\partial B_1\times(-h/2,h/2);\ \int_h^1 \left(\left|\frac{\partial u_h}{\partial r}(r\theta,z)\right|^2 - 1\right)^2 r\, dr \leq h^2|\log_2 h|^2\right\}. \tag{3.5}$$

In addition, it follows, as in the proof of Theorem 1, that

$$\left|\left\{(x',z)\in(B_1\setminus B_{1/2})\times(-1/2,1/2); \left|v_{h,j}(x',z) - |x'|g(x'/|x'|)\right| \geq Ch^{(\gamma-\sigma)/2}\right\}\right| \leq C/|\log_2 h|, \tag{3.6}$$

for $0<\sigma<\gamma<1$ and $2^{-j}\geq h^\sigma$. Next, we claim that, for any $\varepsilon>0$,

$$\frac{1}{h^3}\int_{-h/2}^{h/2}\int_{2^{-j-1}<|x'|<2^{-j}} W(Du_j)\, dx = \frac{1}{\tilde{h}_j^2}\int_{1/2<r<1,\ |z|<1/2} W\left(\nabla' v_{h,j}, \frac{1}{\tilde{h}_j}v_{h,j,3}\right)\, dx \geq E-\varepsilon, \tag{3.7}$$

if $2^{-j} > h^\sigma$ and h is sufficiently small. If (3.7) were false for some $\epsilon>0$, then there would exists $h_j\to 0$, n_j satisfying $2^{-n_j} > h^\sigma$, $\tilde{h}_{j,n_j} = h_j 2^{n_j}$, such that

$$\frac{1}{\tilde{h}_{j,n_j}^2}\int_{1/2<r<1,\ |z|<1/2} W\left(\nabla' v_{h_j,n_j}, \frac{1}{\tilde{h}_{j,n_j}}v_{h_j,n_j,3}\right)\, dx < E-\varepsilon.$$

Applying Theorem 4.1 of [6], we conclude that the rescaled gradients $\left(\nabla' v_{h_j,n_j}, \frac{1}{\tilde{h}_{j,n_j}} v_{h_j,n_j,3}\right)$ are compact in $L^2(B_1 \setminus B_{1/2} \times (-1/2, 1/2))$. Convergence of $\{v_{h_j,n_j}\}$ to $rg(\theta)$ in $L^2(B_1 \setminus B_{1/2} \times (-1/2, 1/2))$ then follows from compactness of the rescaled gradients and the pointwise estimate (3.6). This leads to a contradiction, since it follows, by lower-semicontinuity of the bending energy as given by Theorem 6.1 of [6], that the estimate (3.7) must hold.

The lower bound follows from (3.7) as in the proof of Theorem 1 since $\varepsilon > 0$ and $0 < \sigma < 1$ are arbitrary. □

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