On the equivalence of the static and dynamic asset allocation problems

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Abstract

A classic *dynamic asset allocation problem* optimizes the expected final-time utility of wealth, for an individual who can invest in a risky stock and a risk-free bond, trading continuously in time. This problem was solved by Merton in 1969. Recently, several authors considered the corresponding *static asset allocation problem* in which the individual cannot trade but can invest in options as well as the underlying. The optimal static strategy can never do better than the optimal dynamic one. Surprisingly, however, for some market models the two approaches are equivalent. When this happens the static strategy is clearly preferable, since it avoids transaction costs. This paper examines the question: when, exactly, are the static and dynamic approaches equivalent? We give an easily-tested necessary and sufficient condition, and many nontrivial examples. Our analysis assumes that the stock follows a scalar diffusion process, and uses the completeness of the resulting market model.

1 Introduction

The optimization of asset allocation is a central task of modern finance. A classic model is the

Dynamic Asset Allocation Problem: *Consider an individual who can invest in a risky stock and a risk-free bond, trading continuously in time. Suppose the stock price follows a known diffusion process* $dS = \mu(S,t)S\,dt + \sigma(S,t)S\,dw$*, and assume there is no consumption. What trading strategy optimizes the investor's expected final-time utility of wealth?*

Merton found the answer in 1969, using the method of dynamic programming $[21,\ 22]$. Pliska and Cox & Huang gave an alternative route to the same answer in the 80's using the martingale method

[5, 25]. The ideas in these papers have been very influential, giving rise to a huge literature; see e.g. [18]. Recently two important papers [3, 12] considered the corresponding

Static Asset Allocation Problem: *Consider an individual whose investment opportunities include not only a risk-free bond and a risky stock, but also (European-style) derivatives on the stock. However this individual cannot trade – he is a buy-and-hold investor, who assumes an initial portfolio then holds it to maturity. What portfolio maximizes his expected final-time utility of wealth?*

The equilibrium-based analysis by Carr and Madan [3] assumes option prices are determined by a risk-neutral measure. Different investors may have different utility functions, and also different subjective probability distributions for the final-time price of the stock. Carr and Madan determine the optimal buy-and-hold portfolio in a wide variety of situations. Their analysis includes both the "partial equilibrium" problem in which the risk-neutral measure is given, and also the "full equilibrium" problem in which the risk-neutral measure is determined by the interaction of buy-and-hold investors with different utilities and expectations.

Our work is closer to the analysis of Haugh and Lo [12], who assume the stock follows a scalar diffusion $dS = \mu(S,t)S\,dt + \sigma(S,t)S\,dw$. The market is complete, and option prices are determined by the absence of arbitrage. In this case it is natural to compare the dynamic and static problems. By completeness, every option can be replicated by a trading strategy involving just the stock and the risk-free bond. Therefore the optimum achievable by static asset allocation (using options) cannot be better than the optimum achieved by dynamic asset allocation (without options). Indeed, thinking dynamically, the static problem maximizes the expected final-time utility over a restricted class of trading strategies – namely, those that replicate options.

This restricted class of trading strategies is fairly small. Therefore one might expect the static optimum to be much worse than the dynamic one. Surprisingly, that is not always the case. As Haugh and Lo point out, it is even possible for the static and dynamic problems to be *equivalent*. This occurs whenever the final-time wealth of the optimal dynamic strategy (Merton's final time wealth, so to speak; let's call it W_T) is equal to the payoff of a European option. In other words, the static and dynamic problems are equivalent precisely when W_T is path-independent, in the following sense:

Definition 1. *W_T is path independent if and only if there exists f such that $W_t = f(t, S_t)$, for every $t \in [0, T]$.*

This is the case – by direct inspection of Merton's solution – when the stock price is a geometric Brownian motion, i.e. when $dS = \mu S\,dt + \sigma S\,dw$ with μ and σ constant.

When the static and dynamic problems are equivalent or nearly so, the static strategy is clearly preferable. Indeed, it avoids exposure to market frictions such as transaction costs and limited liquidity. Such frictions were ignored in formulating and solving the dynamic asset allocation problem, but their effect can be significant in practice. Thus it is natural to ask the

Question: *For which market models are the static and dynamic problems equivalent?*

This paper provides the answer. Our results include

(a) a simple necessary and sufficient condition for the problems to be equivalent; and

(b) new families of examples where they are indeed equivalent.

Theorem 1 gives our necessary and sufficient condition. It is easy to test, so the condition permits one to determine, for any given stock price model, whether the associated static and dynamic problems are equivalent – without having to solve either one. Our condition also shows that equivalence is the exception, not the rule. But such exceptions do occur! We shall show, among other examples, that if the risk-free rate is r (assumed constant) then the static and dynamic problems are equivalent

(1) when μ and σ are functions of time alone with

$$\frac{\mu(t) - r}{\sigma^2(t)} = \text{constant}$$

(see Section 3.2);

(2) when σ is constant and

$$\mu = a + \frac{1}{(\sigma^2 - 2a)^{-1} + cS^{\frac{2a}{\sigma^2} - 1}}$$

where a and c are constants such that $c > 0$ and $2a < \sigma^2$ (see Section 4.3); and

(3) when μ is constant with $\mu + r > 0$ and

$$\sigma = \sqrt{\frac{\mu + r}{a} + cS^{-a}}$$

where $a > 0$ and $c \geq 0$ (see Section 4.5).

Our analysis uses the martingale method. The derivation of the necessary and sufficient condition is relatively easy. Most of the paper is devoted to simplifying that condition in special cases, and finding examples.

The static problem permits the investor to buy any option. In other words, he can buy an option *with any payoff.* This is of course an idealization, since the only options with liquid markets are puts and calls. The idealization is acceptable, because an arbitrary payoff can be approximated by a portfolio consisting of puts or calls, stock, and the risk-free bond [1, 3, 11, 24]. For most payoffs, just a few puts or calls are needed to give a relatively good approximation [4, 12].

When the static and dynamic problems are equivalent or nearly so, we prefer the static strategy because it avoids exposure to market frictions. When the static optimum is significantly worse, however, the choice is less clear. Perhaps one could choose by modelling the impact of market frictions, for example by considering a version of the dynamic problem that includes the effect of transaction costs (see e.g. [9]).

But why consider just the static and dynamic strategies? Going beyond the static approach, it is natural to consider strategies involving just a little trading. For example, we could permit trading (i) at a well-chosen intermediate time, or (ii) when the stock price hits a well-chosen threshold. These strategies still limit the investor's exposure to market frictions. We consider their optimization in [15, 16].

Our analysis is restricted to stocks that solve diffusion processes. The asset allocation problem is of course also interesting for other market models – e.g. when the stock price is described by a stochastic volatility or jump-diffusion model. Then options cannot be replicated, so they are not redundant for the dynamic investor, and it is natural to include them among the admissible investments. Two recent papers have considered dynamic asset allocation problems of this type [2, 20]. It is natural to ask whether the static and dynamic asset allocation problems can be equivalent in this more general setting. This question is presently open.

We have noted that the static problem amounts to considering the dynamic one with a restricted set of trading strategies: those that replicate options. Thus, asking whether the static and dynamic problems are equivalent amounts to asking whether the optimal trading strategy happens to replicate an option. As Merton notes in [23], optimal asset allocation is not the only source of interesting trading strategies. It is therefore natural to ask: for any given trading strategy, how well can it be approximated by buying and holding an appropriate option? This problem is entirely open.

The paper is organized as follows: We begin, in Section 2, with a brief review of the martingale approach for solving the dynamic asset allocation problem (which we sometimes refer to as Merton's problem). Then, in Section 3, we give our necessary and sufficient condition for equivalence. Theorem 1 gives the condition in its most general form, when the drift μ and the volatility σ are deterministic functions of stock price and time; Theorem 2 examines the special case when μ and σ are functions of t alone; and Theorem 3 considers the case when μ and σ are functions of the stock price alone.

Section 4 looks for examples of market models for which the dynamic and static problems are equivalent. We find numerous classes of examples, including some in which μ is constant and $\sigma = \sigma(S)$, and others in which σ is constant and $\mu = \mu(S)$. We also give some examples in which μ and σ are both functions of S.

For any given market model, it is natural to ask whether the associated static and dynamic problems are equivalent or not. Theorem 1 provides an easy method for answering this question. To demonstrate this technique, we show in Section 5 that the problems are not equivalent when the logarithm of the stock proce follows an Ornstein Uhlenbeck process.

2 The martingale approach

This section establishes basic notation, and reviews the martingale approach to dynamic portfolio optimization. We'll be working throughout the paper with just one stock. Its price, denoted by S follows the Itô process:

$$dS_t = \mu(t, S_t)S_t\,dt + \sigma(t, S_t)S_t\,dB_t,$$

where B is a one-dimensional Brownian motion under the subjective measure, and μ and σ are deterministic functions of t and S_t. The initial stock price S_0 is always positive, and we assume the solution of the SDE satisfies

$$\int_0^T |\mu(t)|\,dt < \infty \qquad \int_0^T \sigma^2(t)\,dt < \infty \tag{1}$$

almost surely. For simplicity, we assume throughout the paper that the interest rate r is constant.

2.1 Conditions for completeness of the market

Our analysis requires that the market be complete. This places certain conditions on μ and σ, r, see e.g. Chapter 1 of [14]. The main conditions involve the market price of risk, defined by

$$\theta_t = \frac{\mu(t, S_t) - r}{\sigma(t, S_t)}.$$

It must satisfy

$$\int_0^T \theta_t^2\,dt < \infty \tag{2}$$

almost surely; moreover the associated local martingale

$$Z_0(t) = \exp\left[-\int_0^t \theta_s\,dB_s - \frac{1}{2}\int_0^t \theta_s^2\,ds\right]$$

must be a martingale. A well-known sufficient condition is the Novikov criterion:

$$E\left[\exp\left(\frac{1}{2}\int_0^T \theta_t^2\,dt\right)\right] < \infty.$$

The martingale Z_0 is the density of the risk-neutral measure with respect to the subjective measure. In particular, the initial value of an option with payoff f at time T is $e^{-rT}E[Z_0(T)f(S(T))]$. For a more detailed discussion, and a proof that the preceding conditions imply completeness, see for example Theorem 6.6 in Chapter 1 of [14].

We shall refer to the conditions summarized above – (1), (2), and the requirement that Z_0 be a martingale – as *the usual conditions*.

Important remark. Throughout this paper, μ and σ have to satisfy *the usual conditions*.

2.2 Characterization of the optimal final-time wealth

We noted in the Introduction that the static and dynamic problems are equivalent if and only if Merton's final time wealth W_T is path-independent. To understand when this happens we need a representation of W_T. A convenient representation is provided by the martingale approach, which we now review as it applies to our relatively simple setting (with just one stock and no consumption).

The state price density. Define the state price density by

$$H(t) = \exp(-rt) * \exp\left(-\int_0^t \theta_s \, dB_s - \frac{1}{2}\int_0^t \theta_s^2 \, ds\right).$$

As noted above, discounting by this process gives the price of a European option. But we can also make a stronger statement, and this is crucial to the martingale method: for any time-T-measurable random variable W, there is a self-financing trading strategy with initial value $E[H(T)W]$ that replicates W at time T.

The martingale approach to Merton's problem. Consider, for any utility function U, the problem

$$\max_{\pi} E[U(W_\pi(T))] \tag{3}$$

where π ranges over all admissible (self-financing) trading strategies with fixed initial wealth W_0, and $W_\pi(T)$ is the time-T wealth achieved by π. (See e.g. Definition 5.3 in Chapter 5 of [17] for a careful discussion, including the definition of an admissible trading strategy.) The martingale approach splits (3) into two sub-problems:

(1) find the optimal final-time wealth by solving

$$\max_{E[H(T)W]=W_0} E[U(W)] \tag{4}$$

over all time-T-measurable random variables W; then

(2) find an admissible trading strategy π that achieves the optimal W identified in step 1.

Since the market is complete, the second subproblem always has a solution. Therefore to find the optimal final-time wealth we need only consider the first subproblem.

Finding the optimal final time wealth. Our goal is to find "Merton's final-time wealth W_T", the optimal W for (4). The following argument is informal but entirely correct; for a more careful treatment see for example Theorem 16 in Chapter 3 of [18].

We use the method of Lagrange multipliers. The Lagrangian corresponding to (4) is

$$L(W, \lambda) = E\left[U(W) - \lambda\left(H(T)W - W_0\right)\right].$$

The optimal W and the Lagrange multiplier λ are characterized by the first order optimality conditions of the Lagrangian. Taking the first variation with respect to λ gives, as usual, the constraint we started with:

$$L_\lambda(W,\lambda) = W_0 - E\left[H(T)W\right] = 0. \tag{5}$$

Taking the first variation with respect to W gives

$$\langle L_W(W,\lambda), \delta W\rangle = E\left[\left(U'(W) - \lambda H(T)\right)\delta W\right] = 0, \tag{6}$$

for every perturbation δW; this implies $U'(W) = \lambda H(T)$. There exists a unique solution of $U'(W) = \lambda H(T)$, namely $W = (U')^{-1}(\lambda H(T))$. This formula gives the optimal W_T. It remains only to specify the Lagrange multiplier λ; it is determined by (6) since

$$W_0 = E\left[H(T)(U')^{-1}(\lambda H(T))\right] := F(\lambda) \Rightarrow \lambda = F^{-1}(W_0).$$

In solving for λ, we have used the fact that the inverse of F exists. One can see that this is true by taking the derivative of F with respect to λ, using the fact that $H(T)$ is positive and the hypothesis that U (being a utility function) is strictly concave. In conclusion: the optimal final time wealth is given by:

$$W_T = (U')^{-1}(F^{-1}(W_0)H(T)). \tag{7}$$

3 Necessary and sufficient conditions for equivalence

This section presents our necessary and sufficient conditions for equivalence of the static and dynamic problems. The conditions are applicable to any stock process for which the usual conditions hold. Our main result is Theorem 1, presented in 3.1.

When the drift and the volatility depend exclusively on either time or stock price, our conditions simplify considerably. These cases are discussed in Subsections 3.2 and 3.3 respectively.

3.1 General conditions for equivalence

Our starting point is the (trivial) observation that the static and dynamic problems are equivalent if and only if Merton's final-time wealth W_T is path-independent. Our goal is thus to understand when the right hand side of (7) is a path-independent function of the final-time stock price S_T for every T. By inspection, this amounts to asking when the state price density $H(t)$ is a path-independent function of S_t for all t.

It will be convenient to work with the logarithm of the stock price

$$P_t = \ln S_t.$$

Clearly $H(t)$ is a path-independent function of S_t if and only if there exists a deterministic function $g(t,P)$ such that $H(t) = g(t,P_t)$ for all t. The following theorem gives a necessary and sufficient condition by identifying g (if it exists) as the solution of a suitable PDE.

Theorem 1. *Assume the market model satisfies the usual conditions summarized in Section 2.1. Then the static and dynamic problems are equivalent if and only if there exists a function $g(t, P_t)$ with $g(0, P_0) = 0$ such that the following relations hold:*

$$\frac{\mu - r}{\sigma^2} = g_P \tag{8}$$

$$\frac{\mu - r}{\sigma^2}\left(\frac{-\mu - r + \sigma^2}{2}\right) = \frac{1}{2} g_{PP}\sigma^2 + g_t. \tag{9}$$

where $g_P = \frac{\partial g}{\partial P}$, $g_{PP} = \frac{\partial^2 g}{\partial P^2}$, and $g_t = \frac{\partial g}{\partial t}$.

Proof. Consider

$$h(t) = \int_0^t \frac{\mu - r}{\sigma}\, dB_s + \frac{1}{2}\int_0^t \left(\frac{\mu - r}{\sigma}\right)^2 ds.$$

Note that $h(0) = 0$. From the definition of h we have

$$dh_t = \frac{\mu - r}{\sigma}\, dB_t + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 dt. \tag{10}$$

As explained above, the static and dynamic problems are equivalent if and only if

$$h(t) = g(t, P_t) \tag{11}$$

for some function g.

Suppose there is such a g. Then we can find an SDE for h by applying Itô's lemma to the right hand side of (11). The SDE for $P = \ln S$ is

$$dP_t = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\, dB_t,$$

so Itô's lemma applied to $g(t, P_t)$ gives

$$dh_t = \left[g_P\left(\mu - \frac{\sigma^2}{2}\right) + \frac{1}{2} g_{PP}\sigma^2 + g_t\right] dt + g_P \sigma\, dB_t. \tag{12}$$

The SDE associated with a diffusion process is unique, so the corresponding terms in (10) and (12) must be identical. The condition that the coefficients of dB_t match is precisely (8). The condition that the dt terms match is

$$g_P\left(\mu - \frac{\sigma^2}{2}\right) + \frac{1}{2} g_{PP}\sigma^2 + g_t = \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2.$$

With the aid of (8), we can rewrite this as

$$\frac{\mu - r}{\sigma^2}\left(\mu - \frac{\sigma^2}{2}\right) + \frac{1}{2}g_{PP}\sigma^2 + g_t = \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2,$$

or equivalently as

$$\frac{1}{2}g_{PP}\sigma^2 + g_t = \frac{1}{2}\left(\frac{\mu - r}{\sigma^2}\right)\left((\mu - r) - 2\left(\mu - \frac{\sigma^2}{2}\right)\right).$$

Thus (9) holds too.

The preceding calculation is reversible. If (8) and (9) hold then the SDE characterizing h is the same as the one solved by $g(t, P_t)$. If in addition $g(0, P_0) = 0$ then the initial conditions match as well, and it follows that $h_t = g(t, P_t)$ for all t. □

Remark. When the static and dynamic problems are equivalent, the proof of Theorem 1 gives a formula for the optimal final-time wealth. Indeed, when $h(T) = g(T, P_T)$ we have $H(T) = e^{-rT}e^{g(P_T,T)}$, so

$$W_T = (U')^{-1}(\lambda e^{-rT}e^{g(P_T,T)}).$$

For many of the examples presented in this paper the function g has a simple, explicit formula.

3.2 Simplified conditions when μ and σ depend only on t

Our necessary and sufficient condition simplifies dramatically when μ and σ depend on time alone. This leads to a simple, explicit condition for path independence of W_T.

Theorem 2. *Suppose the stock price process is $dS_t = \mu(t)S_t dt + \sigma(t)S_t dB_t$ where μ and σ are deterministic functions of time alone. Assume the "usual conditions" hold. Then the static and dynamic problems are equivalent if and only if*

$$\frac{\mu(t) - r}{\sigma^2(t)} = \textit{constant}.$$

Proof. Assume $(\mu(t) - r)/\sigma^2(t)$ is constant, and call its value a. We shall find the associated solution of (8)-(9) explicitly. Remember that in (8) and (9), the derivatives of g are taken with respect to $P = \ln(S)$.

From (8) we have that $g_P = constant = a$ which implies $g = aP + c(t)$. Then from (9) we have

$$a\left[-\frac{\mu}{2} + \frac{-r + \sigma^2}{2}\right] = c'(t) \Rightarrow a\left[-\frac{a\sigma^2 + r}{2} + \frac{-r + \sigma^2}{2}\right] = c'(t)$$

$$\Rightarrow c'(t) = a\left(\frac{\sigma^2}{2}(1 - a) - r\right) \Rightarrow c(t) = a\left(\frac{\sigma^2}{2}(1 - a) - r\right)t + \text{constant}.$$

Using this formula for c, we get $g = aP + a\left(\frac{\sigma^2}{2}(1-a) - r\right)t + \text{constant}$. Using the initial condition $g(0, P_0) = 0$, we find that the constant is $-aP_0$. Thus finally

$$g = \frac{\mu - r}{\sigma^2}(P - P_0) + \frac{\mu - r}{\sigma^2}\left(\frac{-\mu - r + \sigma^2}{2}\right)t.$$

One verifies by inspection that if $(\mu(t)-r)/\sigma^2(t) = a$ then this g does indeed satisfy $g(0, P_0) = 0$ and our conditions (8) and (9). Thus the static and dynamic problems are equivalent in this case.

Conversely, if the static and dynamic problems are equivalent there must be a function g which satisfies (8) and (9). Because μ and σ are functions of t alone, (8) implies

$$g = \frac{\mu - r}{\sigma^2}P + A(t).$$

But then $g_{PP} = 0$ and $g_t = \left(\frac{\mu - r}{\sigma^2}\right)' P + A'(t)$. Since the left side of (9) depends only on t this implies that $\left(\frac{\mu - r}{\sigma^2}\right)' = 0$. Thus $\frac{\mu - r}{\sigma^2}$ is constant, as asserted. □

Haugh and Lo observed in [12] that the static and dynamic problems are equivalent when the stock price process is geometric Brownian motion (with constant drift and volatility). Theorem 2 can be viewed as a generalization of this result.

3.3 Simplified conditions when μ and σ depend only on S

Our condition also simplifies considerably when μ and σ depend only on the stock price. The result is not as dramatic as Theorem 2: we cannot immediately characterize all solutions. But in a sense the result is richer: there are, in fact, many solutions, as we shall show in Section 4.

Theorem 3. *Suppose the stock price is $dS_t = \mu(S_t)S_t\,dt + \sigma(S_t)S_t\,dB_t$ where μ and σ are deterministic functions of S alone. Assume it satisfies the usual conditions summarized in Section 2.1. Then the static and dynamic problems are equivalent if and only if*

$$(\mu - r) - (\mu - r)' + 2(\mu - r)\frac{\sigma'}{\sigma} - \frac{\mu^2 - r^2}{\sigma^2} = \textit{constant}, \tag{13}$$

where the derivatives of μ and σ are taken with respect to P=$\ln S$.

Proof. If the static and dynamic problems are equivalent then by Theorem 1 there exists a function g which satisfying (8) and (9). We have

$$\frac{\mu - r}{\sigma^2} = g_P \Rightarrow g(P, t) = \int_{P_0}^{P} \frac{\mu - r}{\sigma^2}\,dx + g(t, P_0).$$

The other derivatives of g can be calculated from this:

$$g_{PP} = \frac{(\mu - r)'\sigma^2 - 2(\mu - r)\sigma\sigma'}{\sigma^4}, \quad g_t = \int_{P_0}^{P} \frac{d}{dt}\left(\frac{\mu - r}{\sigma^2}\right) dx + g_t(t, P_0)$$

Since μ and σ are functions of P alone, we conclude that $g_t = g_t(t, P_0)$. Now (9) gives

$$\frac{\mu - r}{\sigma^2}\left[\frac{\sigma^2 - (\mu + r)}{2}\right] = \frac{1}{2}(\mu - r)' - (\mu - r)\frac{\sigma'}{\sigma} + g_t(t, P_0),$$

or in other words

$$g_t(P_0, t) = \frac{\mu - r}{2} - \frac{\mu^2 - r^2}{2\sigma^2} - \frac{1}{2}(\mu - r)' + (\mu - r)\frac{\sigma'}{\sigma} =: A(P).$$

It follows that $g(t, P_0) = A(P)t +$ constant. But $g(t, P_0)$ cannot depend on P. Therefore $A(P)$ must be constant. Using the definition of A and simplifying a bit, this gives

$$(\mu - r) - (\mu - r)' + 2(\mu - r)\frac{\sigma'}{\sigma} - \frac{\mu^2 - r^2}{\sigma^2} = \text{constant},$$

proving the necessity of condition (13).

Conversely, suppose the stock price follows $\mu = \mu(S)$ and $\sigma = \sigma(S)$ satisfy

$$(\mu - r) - (\mu - r)' + 2(\mu - r)\frac{\sigma'}{\sigma} - \frac{\mu^2 - r^2}{\sigma^2} = c \tag{14}$$

for some constant c. Consider the function

$$g(P, t) = \int_{P_0}^{P} \frac{\mu - r}{\sigma^2}\, dx + \frac{c}{2}t.$$

Obviously $g_P = (\mu - r)/\sigma^2$ so (8) is satisfied. We also have:

$$g_{PP} = \frac{(\mu - r)'\sigma^2 - 2(\mu - r)\sigma\sigma'}{\sigma^4} = \frac{(\mu - r)'}{\sigma^2} - \frac{2(\mu - r)\sigma'}{\sigma^3} \quad \text{and} \quad g_t = \frac{c}{2}$$

Combining these formulas with (14) one easily verifies that

$$\frac{\mu - r}{\sigma^2}\left[\frac{\sigma^2 - (\mu + r)}{2}\right] = \frac{1}{2}\sigma^2 g_{PP} + g_t,$$

i.e. (9) holds. We conclude from Theorem 1 that the static and dynamic problems are equivalent as asserted. □

4 Some examples of equivalence

This section gives examples of market models for which the static and dynamic problems are equivalent. We focus on the case $\mu = \mu(S)$, $\sigma = \sigma(S)$, so our main tool is Theorem 3. Our task is thus to find examples of μ and σ for which (13) holds. They must also satisfy "the usual conditions," i.e. the SDE must have a global-in-time solution and the associated market must be complete.

We begin with an obvious example, followed by an easy one. Then we turn in Sections 4.3–4.5 to examples with constant volatility or constant drift. Finally in Section 4.6 we discuss some examples with nonconstant drift and volatility.

4.1 A trivial example

When $\mu = r$, in other words when

$$dS = rS\,dt + \sigma(S,t)S\,dB, \tag{15}$$

the static and dynamic problems are equivalent for any $\sigma = \sigma(S,t)$ satisfying the "usual conditions." This is a consequence of Theorem 1, taking $g = 0$. Actually it is quite obvious: in this case the risk-neutral measure is the same as the subjective measure, and $\theta = 0$. Our investor is risk-neutral, so he has no incentive to invest in stock. Mathematically: the state price density $H(t) = \exp(-rt)$ is a function of time alone. Thus Merton's final-time wealth W_T, given by (7), is not only path-independent – it is actually a function of T alone, independent of S_T.

4.2 A non-trivial but easy example

The following example is just as easy but less intuitive. The static and dynamic problems are equivalent when $\mu = r + \sigma^2(S,t)$, in other words when the stock price solves

$$dS = (r + \sigma^2(S,t))S\,dt + \sigma(S,t)S\,dB, \tag{16}$$

provided this process satisfies the "usual conditions." This is the case, for example, if $\sigma(S,t)$ is uniformly positive and bounded, since then $\theta = (\mu - r)/\sigma = \sigma$ is bounded as well.

Our assertion is an immediate consequence of Theorem 1, with

$$g(t,P) = (P - P_0) - rt$$

since this g satisfies

$$g_P = 1 = \frac{\mu - r}{\sigma^2} \quad \text{and} \quad \frac{1}{2}g_{PP}\sigma^2 + g_t = -r = \frac{\mu - r}{\sigma^2}\left(\frac{-\mu - r + \sigma^2}{2}\right)$$

when $\mu = r + \sigma^2$.

4.3 Examples with constant volatility

We now give a large class of examples for which σ is constant, namely the processes

$$dS = \left(\frac{1}{(\sigma^2 - 2a)^{-1} + cS^{\left(\frac{2a}{\sigma^2} - 1\right)}} + a \right) S\, dt + \sigma S\, dB \tag{17}$$

where a and c are constants. We claim that every process of the form (17) gives an example, provided it satisfies the usual conditions. These conditions impose some restrictions on a and c, namely

$$2a < \sigma^2 \quad \text{and} \quad c \geq 0;$$

the first condition is needed to avoid blowup at $S \to \infty$, the second to make the drift well-defined for every $S > 0$. With these restrictions on a and c, the SDE (17) has the form $dS = \mu(S)S\, dt + \sigma S\, dB$ with μ a uniformly bounded function of S and σ constant. It follows easily that the usual conditions hold.

The proof that (17) satisfies the conditions of Theorem 3 is a matter of mere algebra. But the reader will surely wonder how we found this class of examples, and whether there might be others. To answer these questions, consider the condition of Theorem 3 with constant σ, in other words the ODE

$$\mu - r - \mu' - \frac{\mu^2 - r^2}{\sigma^2} = \alpha$$

where α is any constant. It is convenient to rewrite this equation as

$$\mu' + \frac{\mu^2}{\sigma^2} - \mu = \beta \tag{18}$$

where $\beta = -r - \alpha + r^2/\sigma^2$ is again an arbitrary constant. We shall show that if $\beta > -\sigma^2/4$ then every solution of (18) has the form

$$\mu = \frac{1}{(\sigma^2 - 2a)^{-1} + cS^{\left(\frac{2a}{\sigma^2} - 1\right)}} + a \tag{19}$$

where a and c are constants. Thus (19) is not the general solution of (18), but rather the most general solution with $\beta > -\sigma^2/4$.

Equation (18) is a Riccati-type differential equation with constant coefficients. A key property of Riccati-type equations is that given a single solution, one can find the most general solution by quadrature [26]. (This fact is elementary, and the application made below is completely self-contained.) Since (18) has constant coefficients, finding a single solution is easy: it suffices to look for a constant solution.

We now implement this program. Fixing the constant β, we look for a constant solution of (18) by solving

$$\frac{a^2}{\sigma^2} - a - \beta = 0. \tag{20}$$

The condition $\beta > -\sigma^2/4$ assures that a solution exists. Actually there are two of them; making a choice, let

$$a = \frac{\sigma^2 - \sigma\sqrt{\sigma^2 + 4\beta}}{2}.$$

Now consider the change of variables

$$z = \frac{1}{\mu - a}.$$

Written as an ODE in z, (20) becomes

$$-z' + z\left(\frac{2a}{\sigma^2} - 1\right) + \frac{1}{\sigma^2} = 0.$$

The general solution is

$$z = -\frac{1}{2a - \sigma^2} + c\exp\left(\left(\frac{2a}{\sigma^2} - 1\right)P\right).$$

Since $\mu = z^{-1} + a$, we are led to (19).

4.4 Examples in the class of CEV processes

A CEV (constant elasticity of variance) process has the form

$$dS_t = \mu S_t\,dt + \sigma_0 S_t^{\alpha+1}\,dB_t \tag{21}$$

where $\mu > 0$ and $\sigma_0 > 0$ are constants and $\alpha < 0$, and the initial condition S_0 is positive. Many authors impose the restriction $-1 < \alpha < 0$ (see e.g. [6, 7]) but others permit $\alpha \leq -1$ (see e.g. [8]); for our purposes any $\alpha < 0$ will do.

We claim that within this class of models, the static and dynamic problems are equivalent if and only if $\mu = r$ or $\mu = -r$. In particular, we get equivalence when

$$dS_t = rS_t\,dt + \sigma_0 S_t^{\alpha+1}\,dB_t \tag{22}$$

or

$$dS_t = -rS_t\,dt + \sigma_0 S_t^{\alpha+1}\,dB_t. \tag{23}$$

The proof is easy. Indeed, for a CEV process $\mu =$ constant and $\sigma(S) = \sigma_0 S^\alpha = \sigma_0\exp(\alpha P)$. Therefore $\mu' = 0$ and $\sigma' = d\sigma/dP = \alpha\sigma_0 S^\alpha$. Thus the necessary and sufficient condition of Theorem 3 says

$$(\mu - r) + 2(\mu - r)\alpha - \frac{\mu^2 - r^2}{\sigma_0^2 S_t^{2\alpha}} = \text{constant}.$$

This is true if and only if $\mu^2 - r^2 = 0$, i.e. when $\mu = r$ or $\mu = -r$.

A subtlety arises whenever one considers the CEV process: the stock price S_t can reach 0 in finite time. We take the convention that if S_t reaches 0 then it remains equal to 0 forever after (the firm has gone bankrupt). With this convention the associated market model is complete. A proof can be found for example in [10]; the essential point is that $\theta = (\mu - r)/\sigma(S) = (\mu - r)\sigma_0^{-1} S^{-\alpha}$ stays bounded as $S \to 0$ since $\alpha < 0$.

In terms of Feller's classification of boundary conditions, the situation at $S = 0$ changes at $\alpha = -1/2$ (see e.g. [8]). Indeed, for $-1/2 \leq \alpha < 0$ the origin is an exit boundary. For $\alpha < -1/2$, on the other hand, the origin is a regular boundary point with a killing boundary condition (corresponding to bankruptcy).

4.5 Examples with constant drift

We already know some examples with constant drift: any process with $\mu = r$ (Section 4.1), geometric Brownian motion (Section 3.2), and a CEV process with $\mu = r$ or $\mu = -r$ (Section 4.4). Generalizing the latter two examples, we now show that the static and dynamic problems are equivalent for any stock process of the form

$$dS = \mu S\, dt + \sqrt{\frac{\mu + r}{a} + cS^{-a}}\, S\, dB \tag{24}$$

with $a > 0$, $c \geq 0$, and $\mu \geq -r$. Notice that (24) reduces to geometric Brownian motion when $c = 0$, and it becomes one of our CEV examples when $\mu = -r$.

As in Section 4.3, the proof that these processes satisfy the conditions of Theorem 3 is a matter of mere algebra. But the reader will wonder how we found this example, and whether there might be others. To answer these questions, consider the condition of Theorem 3 with constant μ, in other words the ODE

$$(\mu - r) + 2(\mu - r)\frac{\sigma'}{\sigma} - \frac{\mu^2 - r^2}{\sigma^2} = \alpha$$

where α is any constant. With the change of variable $f = \sigma^2$ this becomes

$$f'(\mu - r) + f(\mu - r - \alpha) - (\mu^2 - r^2) = 0. \tag{25}$$

Holding α fixed, this is a linear equation with constant coefficients!

Let's assume that $\alpha \neq \mu - r$. (The case $\alpha = \mu - r$ is different and interesting; we address it at the end of this subsection.) Then the general solution of (25) is

$$f = \frac{(\mu + r)(\mu - r)}{\mu - r - \alpha} + cS^{-\frac{\mu - r - \alpha}{\mu - r}}.$$

This leads to (24) with the substitution $a = (\mu - r - \alpha)/(\mu - r)$. Notice that $a \neq 0$ since $\alpha \neq \mu - r$.

Are the static and dynamic problems equivalent for the market associated with (24)? Theorem 3 says yes, provided the SDE has a solution for all time and the associated market is complete. To avoid explosion as $S \to \infty$ we need $a > 0$, and to avoid the square root becoming undefined we need $c \geq 0$. The volatility explodes as $S \to 0$ like $S^{-a/2}$, but this is not a problem. The situation is similar to case of a CEV process, considered in Section 4.4: we solve the SDE with the convention that if S_t ever reaches 0 then it remains equal to 0 forever (the firm has gone bankrupt). The associated market model is complete, because the market price of risk

$$\theta = (\mu - r)/\sigma(S) = \frac{\mu - r}{\sqrt{\frac{\mu+r}{a} + cS^{-a}}}$$

stays uniformly bounded.

We assumed above that the constant α in (25) was different from $\mu - r$. The situation when $\alpha = \mu - r$ is surprisingly different. In this case the ODE (25) becomes

$$f'(\mu - r) - (\mu^2 - r^2) = 0. \tag{26}$$

The trivial solution $\mu = r$ was considered in Section 4.1. If $\mu \neq r$ then the general solution of (26) is $f = (\mu + r)P + c$. Remembering that $P = \ln S$, this corresponds to the process

$$dS = \mu S\, dt + \sqrt{(\mu + r)\ln(S) + c}\, S\, dB. \tag{27}$$

But now there's a problem. For our theory to apply, the solution of this SDE must exist for all time and the market must be complete. We are uncertain whether this is ever true.

There are some choices of the parameters for which (27) has a global-in-time solution. Indeed, it is sufficient that $\mu > r$ and $(\mu+r)\ln(S_0)+c > 0$. To explain why, let $u = \mu+r$ and $P = \ln(S)$, so the SDE becomes

$$dS = \mu S\, dt + \sqrt{uP + c} S\, dB.$$

A simple application of Itô's lemma gives

$$dP = \left(\left(\mu - \frac{c}{2}\right) - \frac{1}{2}uP\right)\, dt + \sqrt{uP + c}\, dB.$$

Making the further change of variables $Q = uP + c$, applying Itô's lemma again, and noticing that $u > 0$ (since $\mu > r \geq 0$) we arrive at the SDE

$$dQ = \left(u\mu - \frac{1}{2}uQ\right)\, dt + u\sqrt{Q}\, dB.$$

Now, from the theory of square root processes (see for example Proposition 6.2.4 [19]), Q does not reach 0 in finite time provided $u\mu \geq u^2/2$. The restriction $\mu > r$ was imposed to make this true.

Unfortunately, we do not know whether the restrictions considered in the last paragraph imply completeness of the market. The answer is not obvious, because the market price of risk $\theta = (\mu - r)/\sigma = (\mu - r)/\sqrt{Q}$ is not uniformly bounded, since the square-root process Q can get arbitrarily close to 0 (though for $\mu > r$ and $Q_0 = (\mu + r)\ln(S_0) + c > 0$ it never reaches 0). Therefore it is not clear (at least, not to us) whether the market model (27) ever satisfies the "usual conditions."

4.6 Examples with nonconstant drift and volatility

We have shown the existence of many examples with constant drift or volatility. What about examples for which both $\mu(S)$ and $\sigma = \sigma(S)$ are functions of S? The processes considered in Section 4.2 fall in this category. We show here how the method of Section 4.3 can be used to give additional examples.

Here's the plan. Recall the necessary and sufficient condition provided by Theorem 3, equation (13), which we repeat here:

$$(\mu - r) - (\mu - r)' + 2(\mu - r)\frac{\sigma'}{\sigma} - \frac{\mu^2 - r^2}{\sigma^2} = c \tag{28}$$

where c is any constant. Holding the volatility function σ and the constant c fixed, we can view this as an ODE to be solved for μ. Since it is a Riccati equation (i.e. it is quadratic in μ), knowing one solution permits us to find the most general solution by quadrature.

There is one catch: we must know a solution to get started. In Section 4.3 we used a constant solution. Here we use our "trivial" and "easy" solutions, namely the ones obtained by taking $\mu = r$ and $\mu = r + \sigma^2$. They correspond to different values of the constant c. Turning the crank, we'll get expressions for the general solution of the Riccati equation (28) for these two values of c.

There is also another catch: Theorem 3 applies only if the SDE has a global solution and the market is complete. This undoubtedly places some restrictions on $\sigma(S)$. Thus we do not claim that the static and dynamic problems are equivalent for all the "examples" obtained below. Rather, we claim that they are equivalent if the associated market is complete.

Examples with $c = 0$. We observed in Section 4.1 that the static and dynamic problems are equivalent when $\mu = r$. Correspondingly, the constant function $\mu = r$ is a solution of (28) with $c = 0$. Using this particular solution, we proceed as in Section 4.3 to find the most general solution of (28) with $c = 0$. Consider the change of variables

$$z = \frac{1}{\mu - r}.$$

After the substitution $\mu = z^{-1} + r$ our ODE becomes

$$\frac{1}{z} + \frac{z'}{z^2} + 2\frac{1}{z}\frac{\sigma'}{\sigma} - \frac{1}{\sigma^2}\frac{1}{z}\left(\frac{1}{z} + 2r\right) = 0 \Rightarrow z' + z + 2\frac{\sigma'}{\sigma}z - \frac{1}{\sigma^2} - \frac{2r}{\sigma^2}z = 0.$$

We thus obtain the following ODE for z:

$$z' + z\left(1 + 2\frac{\sigma'}{\sigma} - \frac{2r}{\sigma^2}\right) - \frac{1}{\sigma^2} = 0.$$

The general solution is

$$z = \frac{\int_{P_0}^{P} \frac{1}{\sigma^2(\zeta)} \exp\left(-\int_{P_0}^{\zeta} 1 - \frac{2r}{\sigma^2} + 2\frac{\sigma}{\sigma'}\right) d\zeta + c_1}{\exp\left(-\int_{P_0}^{P} 1 - \frac{2r}{\sigma^2} + 2\frac{\sigma'}{\sigma}\right)} \tag{29}$$

where c_1 is any constant. The general solution of (28) with $c = 0$ is $\mu = z^{-1} + r$, where z is given by (29).

Examples with $c = -2r$. We observed in Section 4.2 that the static and dynamic problems are equivalent when $\mu = r + \sigma^2$. Correspondingly, the function $\mu = r + \sigma^2$ is a solution of (28) with $c = -2r$. Using this particular solution, we proceed as above to find the most general solution of (28) with $c = -2r$. Let $\mu_0 = r + \sigma^2$ and consider the change of variables

$$z = \frac{1}{\mu - \mu_0} = \frac{1}{\mu - r - \sigma^2}.$$

After the substitutions $\mu = z^{-1} + r + \sigma^2$ and $\mu' = -z^{-2}z' + 2\sigma\sigma'$ the ODE (28) with $c = -2r$ becomes

$$\frac{1}{z} + \sigma^2 + \frac{z'}{z^2} - 2\sigma\sigma' + 2\left(\frac{1}{z} + \sigma^2\right)\frac{\sigma'}{\sigma} - \frac{\left(\frac{1}{z} + \sigma^2\right)\left(\frac{1}{z} + 2r + \sigma^2\right)}{\sigma^2} = -2r.$$

This simplifies to the following ODE for z:

$$z' - z\left(1 + \frac{2r}{\sigma^2} - 2\frac{\sigma'}{\sigma}\right) - \frac{1}{\sigma^2} = 0.$$

The general solution is

$$z = \frac{\int_{P_0}^{P} \frac{1}{\sigma^2(\zeta)} \exp\left(-\int_{P_0}^{\zeta} 1 + \frac{2r}{\sigma^2} - 2\frac{\sigma}{\sigma'}\right) d\zeta + c_1}{\exp\left(-\int_{P_0}^{P} 1 + \frac{2r}{\sigma^2} - 2\frac{\sigma'}{\sigma}\right)} \tag{30}$$

where c_1 is any constant. The general solution of (28) with $c = -2r$ is $\mu = z^{-1} + r + \sigma^2$, where z is given by (29).

5 Proving non-equivalence

Theorems 1 and 3 give necessary and sufficient conditions for the static and dynamic problems to be equivalent. Those conditions are easy to check, making it easy to tell whether equivalence holds or not for any given stock price model.

To show how this works, let's demonstrate that when $P = \ln(S)$ solves an Ornstein-Uhlenbeck process

$$dP_t = -\delta(P_t - \alpha)\,dt + \sigma\,dB_t \tag{31}$$

the static and dynamic problems are not equivalent. (This observation is due to Haugh and Lo [12]. They proved it by finding the optimal expected utility in both the static and dynamic settings, and observing that the answers were different.)

By Itô's lemma (31) is equivalent to

$$\frac{dS_t}{S_t} = \left(-\delta(\ln(S_t) - \alpha) + \frac{\sigma^2}{2}\right)\,dt + \sigma\,dB_t.$$

Therefore the case under consideration is

$$\mu(S) = -\delta(\ln(S) - \alpha) + \sigma^2/2, \quad \sigma = \text{constant}.$$

The condition provided by Theorem 3 is

$$(\mu - r) - (\mu - r)' + 2(\mu - r)\frac{\sigma'}{\sigma} - \frac{\mu^2 - r^2}{\sigma^2} = \text{constant},$$

where the derivatives are with respect to $P = \ln S$. In the present setting $(\mu - r)'$ is constant and σ' vanishes, so the condition becomes

$$\mu - \frac{\mu^2}{\sigma^2} = \text{constant}.$$

This is obviously not true, since μ is linear in P.

6 Discussion

We have shown that while the static and dynamic asset allocation problems are generically different, they coincide in a remarkable number of special cases.

This result is interesting because the static approach avoids transaction costs. Therefore it is practically superior whenever the two problems are mathematically equivalent or nearly so.

Our method has the unfortunate limitation of giving only a "yes-no" answer. It tells us whether the problems are equivalent; however if the answer is negative, it gives no information about the magnitude of the difference. Perhaps with further work such information could be extracted.

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