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Another Thin-Film Limit of Micromagnetics

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Abstract

We consider the variational problem of micromagnetics for soft, relatively small thin films with no applied magnetic field. In terms of the film thickness t, the diameter l and the magnetic exchange length w, we study the asymptotic behavior in the small-aspect-ratio limit $t/l \to 0$, when either (a) $w^2/l^2 \gg (t/l)|\log(t/l)|$ or (b) $w^2/l^2 \sim (t/l)|\log(t/l)|$. Our analysis builds on prior work by Gioia & James and Carbou. The limiting variational problem is much simpler than 3D micromagnetics; in particular it is two-dimensional and local, with no small parameters. The contribution of shape anisotropy reduces, in this limit, to a constant times the boundary integral of $(m \cdot n)^2$.

1. Introduction

The main theoretical tool for studying magnetic microstructure is the micromagnetic variational principle. It is a non-convex, nonlocal variational problem whose relative minima represent the stable magnetization patterns of a ferromagnetic body.

Our focus is on soft thin-film elements. Such ferromagnets – for example, permalloy thin films 10 to 100 nanometers thick and 0.1 to 100 microns in diameter – are relatively easy to manufacture. They are used in many devices, and have been explored at length experimentally.

For sufficiently small thin films, numerical simulations are now routinely used to explore the energy landscape. But simulations are simply experiments. To understand and interpret them, it is natural to do analysis as well.

For sufficiently large thin films, numerical simulation is impractical due to the problem's nonlocal character and the presence of multiple length scales. Analysis is difficult too, but it is the only tool available.

What kind of analysis? Since we are considering thin films, the natural goal is a two-dimensional "asymptotic variational problem" – preferably one that is local, with no small parameters. This paper identifies a parameter range where such a limit exists, and examines it with full mathematical rigor using the framework of Γ-convergence. Our regime involves elements small enough that "flux closure" is not achieved – in particular, $m \cdot n \neq 0$ at the boundary. Our main achievement is to characterize the leading-order magnetostatic energy. The answer turns out to be local and very simple – a constant times $\int_{\partial\omega}(m \cdot n)^2$.

To state our results and link them with prior work, we start by observing that the problem has three distinct parameters with the dimensions of length:

$$\begin{aligned} t &= \text{the film thickness;} \\ l &= \text{the in-plane diameter; and} \\ w &= \text{the exchange length of the ferromagnetic material.} \end{aligned}$$

(See Section 2 for more about micromagnetics, and Section 3 for elaboration of the following discussion.) Therefore there are two nondimensional parameters,

$$h = \frac{t}{l} = \text{aspect ratio}, \qquad d = \frac{w}{l} = \text{normalized exchange length},$$

and a variety of different thin-film regimes, depending on the relation between h and d as $h \to 0$.

Gioia and James [10] considered the behavior when $h \to 0$ with d held fixed. They showed that the magnetostatic energy becomes, to leading order, a penalization term for the out of plane component of the magnetization. In the limit it simply imposes the constraint $m_3 = 0$. This was, to our knowledge, the first rigorous analysis of a micromagnetic thin-film limit. The limit of micromagnetic dynamics in the same regime ($h \to 0$, $d =$ constant) was examined by Ammari et al. [4] and Carbou [5].

In the absence of crystalline anisotropy, and without an applied field, the asymptotic problem of Gioia and James minimizes $\int_\omega d^2|\nabla m|^2 + m_3^2$ subject to $|m| = 1$, where ω is the 2D cross-section of the film. The minimum value is 0, achieved by any constant m with $m_3 = 0$. Carbou observed that this degeneracy is broken by the next-order term in the magnetostatic energy [5]. He showed, moreover, that *when evaluated at constant magnetizations* it equals a constant times $\int_{\partial\omega}(m \cdot n)^2$.

A very different regime was considered recently by DeSimone, Kohn, Müller and Otto [8]. They studied the behavior when $h \to 0$ and $d \to 0$ with $d^2 \ll h/|\log h|$. This regime corresponds to a thin but relatively large film. The asymptotic variational problem is nonlocal and degenerate, but convex; moreover it uniquely determines the "magnetic charges" associated with nonzero $\operatorname{div} m$ or $m \cdot n$. In the absence of crystalline anisotropy and without an applied field, the asymptotic problem minimizes $\|\operatorname{div} m\chi_\omega\|^2_{H^{-1/2}}$ subject to $m_3 = 0$ and $|m| \leq 1$. The minimum value is 0, achieved for example at $m = 0$. Thus, in the regime of [8] flux closure is always achieved, i.e. $\operatorname{div} m = 0$ in ω and $m \cdot n = 0$ at $\partial\omega$ when there is no applied field.

The present paper considers a regime that's intermediate between those of [5, 10] and [8]. Our main focus is the case when $h \to 0$ and $d^2 = \alpha h |\log h|$ with $0 < \alpha < \infty$. (We also consider the case $d^2 \gg h|\log h|$, which corresponds to $\alpha = \infty$ and includes the Gioia-James regime.) If the material parameter w and the aspect ratio h are held fixed, then our focus is upon elements somewhat larger than the ones considered by Gioia & James but much smaller than those of DeSimone et al. Our limit problem minimizes

$$\alpha \int_\omega |\nabla m|^2 + \frac{1}{2\pi} \int_{\partial\omega} (m \cdot n)^2$$

subject to $m_3 = 0$ and $|m| = 1$. Unlike the setting of Gioia-James and Carbou, the optimal m is not constant. But the magnetostatic energy is still asymptotically a constant times $\int_{\partial\omega} (m \cdot n)^2$. In particular, the asymptotic problem is local and relatively simple. The boundary integral expresses the effect of "shape anisotropy" in this regime.

Other intermediate regimes are possible. Ours is distinguished by the fact that it prohibits walls and vortices. Other limits which permit interior or boundary vortices are being considered by Kurzke [12] and Moser [13]; see Section 3 for a unifying discussion.

Our analysis is fully rigorous. Following the lead of Gioia & James, we work in rescaled variables, formulating the 3D variational problems on a fixed cylinder (independent of h and d). The asymptotic variational problem is obtained as a Γ-limit. Our analysis of the magnetostatic term draws from the argument of Carbou. We must work more, however, because m is not assumed to be constant, or even smooth. Can the magnetostatic energy associated with nonzero $m \cdot n$ be somehow reduced by making m vary rapidly in a boundary layer near $\partial\omega$? The answer is no – but this is a consequence of our analysis rather than a hypothesis.

The paper is organized as follows. Section 2 introduces the variational problem of micromagnetics. Section 3 explains heuristically our choice of regime, and discusses its relation to those studied by DeSimone et al, Gioia & James, Kurzke, and Moser. Section 4 is the mathematical core of the paper: Section 4.1 states our main results, which assert Γ-convergence of the (rescaled) 3D micromagnetic problems to a suitable 2D problem; Section 4.2 studies the asymptotics of the magnetostatic energy; and Section 4.3 applies that analysis to prove our main results. The Appendices present material that is well-known but difficult to find in the literature: Appendix A supports Section 4.2 by proving some estimates, while Appendix B supports Section 3 by examining the magnetostatic energy when m is independent of the thickness.

2. Basic micromagnetics

The micromagnetic variational principle uses classical physics to explain the magnetization distribution in a ferromagnetic sample. Its local minima are the stable (hence observable) magnetization distributions. First developed in the 1930's by Brown and Landau & Lifshitz, micromagnetics has been shown to capture the remarkable multiscale complexity of magnetic behavior.

After a suitable normalization, the micromagnetic energy has the form

$$\mathcal{E}(m) = w^2 \int_\Omega |\nabla m|^2 + Q \int_\Omega \phi(m) + \int_{\mathbf{R}^3} |\nabla u|^2 - 2 \int_\Omega h_{ext} \cdot m. \tag{1}$$

The four terms are known as the exchange, anisotropy, magnetostatic, and external (or Zeeman) energies respectively. The domain $\Omega \subset \mathbf{R}^3$ is the region occupied by the ferromagnet; $m\colon \Omega \to \mathbf{R}^3$ is the normalized magnetization, constrained by

$$|m(x)| = 1 \quad \text{for } x \in \Omega \tag{2}$$

and understood to equal 0 outside Ω; and h_{ext} is the external, applied field. The function u, defined on all $\mathbf{R}^3$, is defined by

$$\operatorname{div}(\nabla u + m) = 0 \quad \text{in } \mathbf{R}^3, \tag{3}$$

in the sense of distributions. Thus the magnetostatic energy is nonlocal in m. Its physical interpretation involves the long-range interaction of magnetic dipoles; mathematically it can be viewed as a penalization favoring $\operatorname{div} m = 0$, since ∇u is the Helmholtz projection of m onto gradients. It can be expressed as an integral over Ω alone, since

$$\int_{\mathbf{R}^3} |\nabla u|^2 = -\int_\Omega m \cdot \nabla u = \int_\Omega h_{ind} \cdot m,$$

where

$$h_{ind} = -\nabla u$$

is the magnetic field induced by m. The nonlocal character of this term makes it onerous to evaluate, and a major stumbling block to numerical simulation.

To explain a bit more, we comment on each term separately:

- The anisotropy energy $\phi(m)$ favors special directions of the magnetization. Its coefficient Q is a nondimensional material parameter controlling the relative importance of anisotropy versus magnetostatic energy. Magnetic thin-film devices are usually made from "soft" materials, i.e. those for which Q is small and crystalline anisotropy plays a minor role. Our focus is on this case.

– The magnetostatic energy $|\nabla u|^2$ favors $\operatorname{div} m = 0$ in Ω and $m \cdot n = 0$ at $\partial\Omega$, where n is the unit normal to $\partial\Omega$. In thin-film limits this term always produces a constraint $m_3 = 0$. It may or may not also force $m \cdot n = 0$ at the film edge, depending on the choice of scaling regime.
– The exchange energy $|\nabla m|^2$ penalizes spatial variation of m. Its coefficient w is typically on the order of 10 nanometers. In extremely small magnets this term dominates, making the magnetization approximately constant.
– The external (Zeeman) energy favors magnetization parallel to an externally applied field. This term is very important in practice, since a thin-film element is "switched" by applying a suitable in-plane magnetic field. The relevant applied fields are, of course, those large enough to have an effect but small enough to interact with other terms. Thus h_{ext} should be scaled so that the Zeeman energy interacts with the leading-order behavior of exchange and/or magnetostatic energy.

For more information on micromagnetics see [1], [11].

We shall for simplicity take $Q = 0$ and $h_{ext} = 0$. It is trivial, however, to include a little anisotropy and an appropriately-scaled Zeeman term, since Γ-convergence is insensitive to compact perturbations of the functional.

It is convenient to nondimensionalize lengths by replacing (x, y, z) with $(x', y', z') = (x/l, y/l, z/l)$, $m'(x', y', z') = m(x, y, z)$, and $u'(x', y', z') = l^{-1}u(x, y, z)$. Dropping the primes, and taking $Q = h_{ext} = 0$, we have $\mathcal{E} = l^3 E$ with

$$E(m) = d^2 \int_{\Omega_h} |\nabla m|^2 + \int_{\mathbf{R}^3} |\nabla u|^2. \tag{4}$$

Here $\Omega_h = \omega \times (0, h)$, where $\omega \subset \mathbf{R}^2$ is the rescaled cross-section (with diameter one); $d = w/l$; $h = t/l$; $m = (m_1, m_2, m_3)$ satisfies $|m| = 1$ on Ω_h and is extended by 0 off Ω_h; and u_h solves

$$-\Delta u = \operatorname{div} m \quad \text{in } \mathbf{R}^3. \tag{5}$$

This nondimensionalization does not of course change the essential problem. Nor does the simplification $Q = h_{ext} = 0$. The functional (4) is still a non-convex, non-local variational problem, with all the difficulties of the full micromagnetic model.

3. Heuristics and scalings

This section explains our choice of scaling, and places it into the context of other related work.

The main issue is the asymptotic behavior of the magnetostatic term. To explain it, we assume – *for this section only* – that $m = (m_1, m_2, m_3)$ is independent of the thickness variable z, and that it does not change significantly as h and d vary. (Our rigorous analysis, presented in Section 4, makes no such assumptions.)

Let m be defined on $\Omega_h = \omega \times (0, h)$, as in (4)–(5), extended by 0 off Ω_h. We denote the in-plane component of the magnetization by

$$m' = (m_1, m_2),$$

and for any function $f(x)$ we denote its Fourier transform by

$$\hat{f}(\eta) = \int f(x) e^{-2\pi i x \cdot \eta}.$$

If m is independent of z, the associated ∇u can be determined by separation of variables, or equivalently by using the 3D Fourier transform. One finds that

$$\int_{\mathbf{R}^3} |\nabla u|^2 = h \int_{\mathbf{R}^2} \frac{(\eta \cdot \hat{m}')^2}{|\eta|^2} \left(1 - \hat{\Gamma}_h(|\eta|)\right) + h \int_{\mathbf{R}^2} \hat{m}_3^2 \hat{\Gamma}_h(|\eta|), \tag{6}$$

where $\eta = (\eta_1, \eta_2)$ and

$$\hat{\Gamma}_h(|\eta|) = \frac{1 - \exp(-2\pi h|\eta|)}{2\pi h|\eta|}. \tag{7}$$

We note for later reference that the inverse Fourier transform of $\hat{\Gamma}_h$ is

$$\Gamma_h(x) = \frac{1}{2\pi h} \left(\frac{1}{|x|} - \frac{1}{\sqrt{|x|^2 + h^2}} \right). \tag{8}$$

Formulas (6)–(8) are well-known and may be found in [2], [9]. For completeness of the presentation the calculation is sketched in Appendix B, following [9].

Since $\hat{\Gamma}_h \to 1$ as $h \to 0$, the second term of (6) is asymptotically

$$E_{\text{trans}} = h \int_\omega m_3^2.$$

The first term of (6) represents the energy due to lack of flux-closure, i.e. due to a nonzero in-plane divergence $\operatorname{div}_p m$ in ω and/or a nonzero normal component $m \cdot n$ at $\partial\omega$. It is important to separate these two sources, because their energies scale differently. Since $1 - \hat{\Gamma}_h \to \pi h \eta$ as $h \to 0$, the contribution due to nonzero (but smooth) $\operatorname{div}_p m$ is asymptotically

$$E_{\text{bulk}} = h^2 \|(\operatorname{div}_p m)_{\text{smooth}}\|^2_{H^{-1/2}}$$

(up to a constant, which can be absorbed into the definition of the $H^{-1/2}$ norm). The contribution of nonzero $m \cdot n$ is different, because the associated contribution to $\operatorname{div}_p m$ is singular – a measure concentrated on $\partial\omega$. We know from Carbou [5] that when m is uniform this term is asymptotically a constant times the boundary integral of $(m \cdot n)^2$. We shall show in Section 4 that the same result holds more generally: the contribution of nonzero $m \cdot n$ is asymptotically

$$E_{\text{bdry}} = \frac{1}{2\pi} h^2 |\log h| \int_{\partial\omega} (m \cdot n)^2.$$

The exchange energy is local, hence easy to handle: when m is independent of z it becomes

$$E_{\text{exch}} = d^2 h \int_\omega |\nabla m|^2.$$

Thus the energy (4) is formally

$$\begin{aligned} E &= E_{\text{exch}} + E_{\text{bulk}} + E_{\text{bdry}} + E_{\text{trans}} && (9) \\ &= d^2 h \int_\omega |\nabla m|^2 + h^2 \|(\operatorname{div}_p m)_{\text{smooth}}\|^2_{H^{-1/2}} && (10) \\ &+ \frac{1}{2\pi} h^2 |\log h| \int_{\partial\omega} (m \cdot n)^2 + h \int_\omega m_3^2. \end{aligned}$$

Our focus in this paper is on two slightly different regimes. In the first, E_{trans} and E_{exch} become constraints, E_{bdry} is the leading-order term, and E_{bulk} is negligible. In the second, E_{trans} becomes a constraint, E_{exch} and E_{bdry} interact as leading-order terms, and E_{bulk} is negligible. Specifically:

- The regime $d^2 h \gg h^2 |\log h|$ includes the case considered by Gioia & James and Carbou. It corresponds to extremely small elements, of thickness around 1nm and diameter around 10nm. The minimum energy is of order $h^2|\log h|$, and the optimal (constant) magnetization solves the asymptotic variational problem

$$\min_{\substack{|m|=1,\, m_3=0 \\ m=\text{constant}}} \int_{\partial\omega} (m \cdot n)^2. \tag{11}$$

  Usually (11) selects a unique optimal direction for m (up to sign). This effect is sometimes called "shape anisotropy." When ω is sufficiently symmetric, however – for example when it is a square – the functional (11) is degenerate, assigning the same value to every constant magnetization. In experiments, there is still a preferred direction – this effect is called "configurational anisotropy," see e.g. [6]. The conventional view is that one must look for higher-order corrections to the leading-order term (11). An alternative view – more correct, in our opinion – is that the experiments are better modeled by (12).
- The regime $d^2 h = \alpha h^2 |\log h|$ with $0 < \alpha < \infty$ corresponds to somewhat larger elements, for example with thickness around $3-5$nm and diameter around $30 - 50$nm. The minumum energy is again of order $h^2|\log h|$, but now the optimal magnetization solves the asymptotic variational problem

$$\min_{|m|=1,\, m_3=0} \alpha \int_\omega |\nabla m|^2 + \frac{1}{2\pi} \int_{\partial\omega} (m \cdot n)^2. \tag{12}$$

In a square, numerical simulation suggests that the energy-minimizer is always a *leaf state*. When α is large, it appears that the leaf states are the only relative minima of (12). When α is small, however, there are additional local minima known as *buckle states* (see Figure 1).

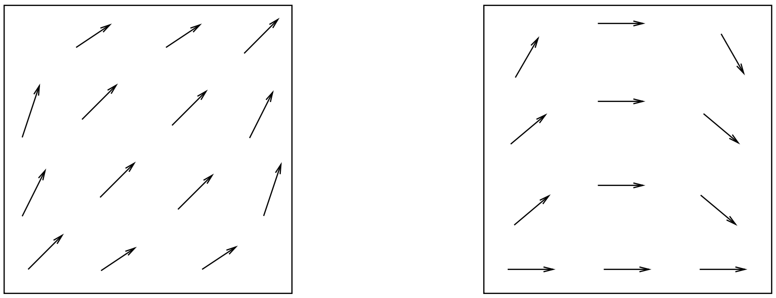

Fig. 1. Sketch of the leaf state (left) and buckle state (right).

Since E_{bulk} and E_{bdry} scale differently by a factor of $|\log h|$, a thin-film limit cannot keep both as leading-order terms. Our regimes keep E_{bdry}, so they necessarily drop E_{bulk}. Recent work by Roger Moser [13] explores the different regime $d^2 = h$. In this case E_{bdry} joins E_{trans} as a constraint, and the asymptotic problem is formally to minimize

$$\int_\omega |\nabla m|^2 + \|(\mathrm{div}_p\, m)_{\mathrm{smooth}}\|^2_{H^{-1/2}} \tag{13}$$

subject to $m_3 = 0$, $|m| = 1$ in ω and $m \cdot n = 0$ at $\partial\omega$. This formal limit is not quite right, however. Indeed, when ω is simply-connected it is impossible for m to satisfy the constraints and have finite exchange energy. What actually happens is that the magnetization develops a vortex in the interior of ω [13].

Recent work by Kurzke [12] considers the limit of (12) as $\alpha \to 0$. This is different from Moser's regime, because it does not involve E_{bulk}; formally, at least, it represents the case $h \ll d^2 \ll h|\log h|$. Naively, the asymptotic problem would appear to be

$$\int_\omega |\nabla m|^2 \tag{14}$$

subject to $m_3 = 0$, $|m| = 1$ in ω and $m \cdot n = 0$ at $\partial\omega$. As in the case of (13), this naive limit is not quite right, because there is no admissible m with finite energy. What actually happens is that the magnetization develops "boundary vortices" along $\partial\omega$ [12]. (The analysis of (12) in the $\alpha \to 0$ limit is certainly an interesting problem. We are not certain, however, that it represents the limit of micromagnetics when $h \to 0$ with $h \ll d^2 \ll h|\log h|$, since our estimates justifying (12) are not uniform in this limit.)

The regime considered by DeSimone et al [8] is quite different from any of these. Defined by $d^2 \ll h/|\log h|$, it has the property that E_{exch} is negligible, while E_{trans} and E_{bdry} become constraints. Therefore the asymptotic problem involves only E_{bulk}.

It can be confusing to think about so many regimes at once. Figure 2 summarizes their regions of validity, for a fixed (sufficiently small) value of the aspect ratio h. The axis variable is $d^2 = (w/l)^2$, so points at the far right correspond to the *smallest* elements, and points at the far left to the *largest* ones.

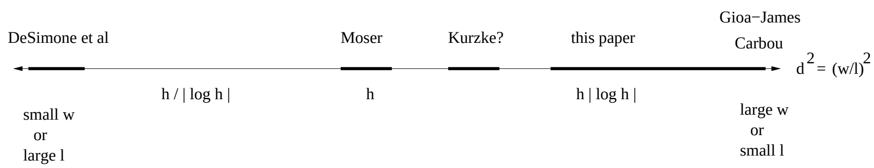

Fig. 2. The various regimes, at fixed aspect ratio h.

As the preceding discussion makes clear, we take the view that h is negligible relative to $h|\log h|$. This is impeccable mathematics but dubious physics, since for soft thin films of practical interest h is rarely smaller than 10^{-5}. Thus for realistic values of h our asymptotic problem (12) should probably be modified by adding E_{bulk}; we doubt this would change its qualitative behavior. A similar modification of (11) is unnecessary, since the admissible magnetizations are constant.

4. The main result

In this section we are going to prove the results announced above. We show that under the scalings $\frac{d^2}{h|\log h|} \to \infty$ and $\frac{d^2}{h|\log h|} \to \alpha$ the full micromagnetic energy functional reduces to relatively simple 2D variational problems where the magnetostatic energy becomes an integral over the boundary. The limiting magnetization m is

- independent of the thickness variable z;
- has no out of plane component m_3;
- keeps the constraint $|m| = 1$.

The proof uses the ideas from the previous section but is more subtle due to the fact that magnetization m may depend on thickness.

4.1. The mathematical problem

Let us state with mathematical precision the problem we will solve. Consider the one parameter family of micromagnetic energy functionals

$$E_h(m_h) = \frac{d^2}{h^2|\log h|} \int_{\Omega_h} |\nabla m_h|^2 + \frac{1}{h^2|\log h|} \int_{\mathbf{R}^3} |\nabla u_h|^2, \qquad (15)$$

where $\Omega_h = \omega \times (0,h)$ ($\omega \subset \mathbf{R}^2$), $|m_h| = 1$ and u_h satisfies the following equation

$$-\Delta u_h = \operatorname{div}(m_h \chi(\Omega_h)) \qquad \text{in } \mathbf{R}^3. \tag{16}$$

Now if we rescale the domain in z direction, we obtain

$$E_h(\tilde m_h) = \frac{d^2}{h|\log h|} \int_\Omega \left(|\nabla' \tilde m_h|^2 + \frac{1}{h^2} (\frac{\partial \tilde m_h}{\partial z})^2 \right) + \frac{1}{h^2 |\log h|} \int_{\mathbf{R}^3} |\nabla u_h|^2, \tag{17}$$

where $\tilde m_h(x,y,z) = m(x,y,hz)$ for $z \in (0,1)$, $\nabla' = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ and $\Omega = \omega \times (0,1)$. Note that magnetostatic energy is written as in (15). This is the functional we are going to consider. We will prove the following two theorems

Theorem 1. *Suppose* $\frac{d^2}{h|\log h|} \to \infty$ *as* $h \to 0$. *Then we have:*

- *if* $E_h(\tilde m_h) \le C$ *then* $\tilde m_h \to m$ *strongly in* $H^1(\Omega; S^2)$ *(maybe for a subsequence),* $m =$*const and* $m_3 = 0$*;*
- $$E_h \to^\Gamma E_1 \qquad \textit{in } H^1(\Omega; S^2),$$

  where

  $$E_1(m) = \begin{cases} \frac{1}{2\pi} \int_{\partial\omega} (m \cdot n)^2 & \textit{if } m = const \textit{ and } m_3 = 0 \\ \infty & \textit{otherwise} . \end{cases} \tag{18}$$

Theorem 2. *Suppose* $\frac{d^2}{h|\log h|} \to \alpha$ *then we have:*

- *if* $E_h(\tilde m_h) \le C$ *then* $\tilde m_h \to m$ *weakly in* $H^1(\Omega; S^2)$ *(maybe for a subsequence),*
  $m = m(x,y)$ *and* $m_3 = 0$*;*
- $$E_h \to^\Gamma E_2 \qquad \textit{in } H^1_w(\Omega; S^2),$$

  where

  $$E_2(m) = \begin{cases} \int_\omega \alpha |\nabla m|^2 + \frac{1}{2\pi} \int_{\partial\omega} (m \cdot n)^2 & \textit{if } m = m(x,y) \textit{ and } m_3 = 0 \\ \infty & \textit{otherwise.} \end{cases} \tag{19}$$

Theorem 1 insludes the result derived by Carbou [5] under hypothesis that magnetization is constant. Our analysis does not require any assumptions on m_h. The proofs of these theorems use (a) the explicit form of the magnetostatic energy, expressed in terms of convolution operators; (b) standard estimates for convolution operators. In particular we will use of the following inequalities

Lemma 1 (Generalized Young's inequality). *Assume $\Omega \subset \mathbf{R}^n$ is a bounded set, $f, g \in L^2(\Omega)$ and $K \in L^1_{loc}(\mathbf{R}^n)$. Then*

$$\int_\Omega \int_\Omega f(x)g(y)K(x-y) \leq \|K\|_{L^1(B)}\|f\|_{L^2(\Omega)}\|g\|_{L^2(\Omega)} \tag{20}$$

for some fixed ball $B \subset \mathbf{R}^n$.

Lemma 2. *Assume $\omega \subset \mathbf{R}^2$, $f \in L^2(\omega)$ and $g \in L^2(\partial\omega)$ then*

$$\int_{\partial\omega}\int_\omega \frac{f(x)g(y)}{|x-y|} \leq C\|f\|_{L^2(\omega)}\|g\|_{L^2(\partial\omega)}.$$

See appendix A for proofs.

Before proving the actual Γ- convergence results we want to simplify the magnetostatic energy.

4.2. Calculation of magnetostatic energy.

Here we provide a rigorous explanation of the heuristic formula (9) for the magnetostatic energy used before. We essentially follow the ideas of Carbou [5].

First we prove the following result.

Lemma 3. *Let $\bar{m}_h = \frac{1}{h}\int_0^h m_h(x,z)dz$ and let $\bar{u}_h$ be a solution of (16) with m_h replaced by $\bar{m}_h$. Then the following inequality holds:*

$$\frac{1}{h^2|\log h|}\left|\int_{\mathbf{R}^3}|\nabla u_h|^2 - \int_{\mathbf{R}^3}|\nabla \bar{u}_h|^2\right| \leq \frac{C}{|\log h|}\left(\frac{1}{h^2}\|\frac{\partial \bar{m}_h}{\partial z}\|^2_{L^2(\Omega)} + 1\right). \tag{21}$$

Proof. Using linearity of (16) and integration by parts we obtain

$$\int_{\mathbf{R}^3}|\nabla u_h - \nabla \bar{u}_h|^2 \leq \int_{\Omega_h}|m_h - \bar{m}_h|^2.$$

By Poincaré's inequality applied only for z variable we have

$$\int_{\Omega_h}|m_h - \bar{m}_h|^2 \leq Ch^2\int_{\Omega_h}|\frac{\partial m_h}{\partial z}|^2.$$

Therefore, combining the last two inequalities and using the triangle ineqality we obtain

$$\left|\left(\int_{\mathbf{R}^3}|\nabla u_h|^2\right)^{\frac{1}{2}} - \left(\int_{\mathbf{R}^3}|\nabla \bar{u}_h|^2\right)^{\frac{1}{2}}\right| \leq Ch\|\frac{\partial m_h}{\partial z}\|_{L^2(\Omega_h)}. \tag{22}$$

Using integration by parts we obtain

$$\left(\int_{\mathbf{R}^3}|\nabla u_h|^2\right)^{\frac{1}{2}} \leq \|m_h\|_{L^2(\Omega_h)} \text{ and } \left(\int_{\mathbf{R}^3}|\nabla \bar{u}_h|^2\right)^{\frac{1}{2}} \leq \|\bar{m}_h\|_{L^2(\Omega_h)}. \tag{23}$$

Combining (22) and (23), and using the fact that $|\bar{m}_h| \leq |m_h| = 1$ we have

$$\left| \int_{\mathbf{R}^3} |\nabla u_h|^2 - \int_{\mathbf{R}^3} |\nabla \bar{u}_h|^2 \right| \leq C h^{\frac{3}{2}} \| \frac{\partial m_h}{\partial z} \|_{L^2(\Omega_h)}.$$

Rescaling domain Ω_h to Ω we finally have

$$\left| \int_{\mathbf{R}^3} |\nabla u_h|^2 - \int_{\mathbf{R}^3} |\nabla \bar{u}_h|^2 \right| \leq C h \| \frac{\partial \tilde{m}_h}{\partial z} \|_{L^2(\Omega)}.$$

The conclusion of lemma follows.

In view of lemma 3 we need to estimate only $\int_{\mathbf{R}^3} |\nabla \bar{u}_h|^2$.

$$\begin{aligned} \int_{\mathbf{R}^3} |\nabla \bar{u}_h|^2 &= -\int_{\Omega_h} \nabla \bar{u}_h \cdot \bar{m}_h \\ &= \int_{\Omega_h} \bar{u}_h \operatorname{div} \bar{m}_h - \int_{\partial \Omega_h} \bar{u}_h (\bar{m}_h \cdot n). \end{aligned} \tag{24}$$

Solving the equation for u_h we also have

$$4\pi \bar{u}_h(x) = \int_{\Omega_h} \frac{1}{|x-y|} \operatorname{div} \bar{m}_h(y) - \int_{\partial \Omega_h} \frac{1}{|x-y|} (\bar{m}_h \cdot n)(y). \tag{25}$$

Plugging the expression (25) for $\bar{u}_h$ into formula (24) we obtain

$$\begin{aligned} 4\pi \int_{\mathbf{R}^3} |\nabla \bar{u}_h|^2 &= \int_{\Omega_h} \int_{\Omega_h} \frac{1}{|x-y|} \operatorname{div} \bar{m}_h(y) \operatorname{div} \bar{m}_h(x) \\ &+ \int_{\partial \Omega_h} \int_{\partial \Omega_h} \frac{1}{|x-y|} (\bar{m}_h \cdot n)(y) (\bar{m}_h \cdot n)(x) \\ &- 2 \int_{\partial \Omega_h} \int_{\Omega_h} \frac{1}{|x-y|} \operatorname{div} \bar{m}_h(y) (\bar{m}_h \cdot n)(x). \end{aligned} \tag{26}$$

Below we are going to describe three terms in (26) as "bulk-bulk term", "boundary-boundary term", and "bulk-boundary term". We will expand these terms using the fact that $\bar{m}_h$ is independent of thickness variable and then estimate them. Below we use the following notation:

$x, y \in \omega$ are in-plane variables, $\quad s, t \in [0, h]$ are thickness variables.

Bulk-bulk term.
Bulk-bulk term is equal to

$$A = \int_0^h \int_0^h \int_\omega \int_\omega \frac{\operatorname{div}_p \bar{m}_h(x) \operatorname{div}_p \bar{m}_h(y)}{\sqrt{|x-y|^2 + (s-t)^2}}. \tag{27}$$

Here $\operatorname{div}_p \bar{m}_h$ denotes a plane divergence.

Bulk-boundary term.
Bulk-boundary term is equal to

$$B = -2\int_0^h\int_0^h\int_{\partial\omega}\int_\omega \frac{\operatorname{div}_p \bar{m}_h(x)(\bar{m}_h\cdot n)(y)}{\sqrt{|x-y|^2+(s-t)^2}} \tag{28}$$

Boundary-boundary term.
Expanding and reorganizing the boundary-boundary term we find it equal to

$$C = 4\pi h\int_\omega\int_\omega (\bar{m}_h\cdot e_3)(x)\Gamma_h(x-y)(\bar{m}_h\cdot e_3)(y) + \int_{\partial\omega}\int_{\partial\omega}\int_0^h\int_0^h \frac{(\bar{m}_h\cdot n)(x)(\bar{m}_h\cdot n)(y)}{\sqrt{|x-y|^2+(s-t)^2}}, \tag{29}$$

Notice that

$$\Gamma_h(x-y) = \frac{1}{2\pi h}\left(\frac{1}{|x-y|} - \frac{1}{\sqrt{|x-y|^2+h^2}}\right)$$

is the same as in (8).

Now we want to estimate $|A|$ and $|B|$. All the estimates we produce below involve $\tilde{m}$ instead of m. Using the Generalized Young's Inequality (lemma 1) and the fact that

$$|\operatorname{div}_p \bar{m}_h(x)| \le \|\operatorname{div}_p \tilde{m}_h(x)\|_{L^2(0,1)} \tag{30}$$

we have

$$|A| \le h^2\int_\omega\int_\omega \frac{|\operatorname{div}_p \bar{m}_h(x)||\operatorname{div}_p \bar{m}_h(y)|}{|x-y|} \le Ch^2\|\operatorname{div}_p \bar{m}_h\|^2_{L^2(\omega)} \le Ch^2\|\operatorname{div}_p \tilde{m}_h\|^2_{L^2(\Omega)}. \tag{31}$$

Using lemma 2, (30) and the fact that $|\bar{m}_h\cdot n| \le 1$ we obtain

$$|B| \le 2h^2\int_{\partial\omega}\int_\omega \frac{\operatorname{div}_p \bar{m}_h(x)(\bar{m}_h\cdot n)(y)}{|x-y|} \le Ch^2\|\operatorname{div}_p \bar{m}_h\|_{L^2(\omega)}\|\bar{m}_h\cdot n\|_{L^2(\partial\omega)} \le Ch^2(1+\|\operatorname{div}_p \tilde{m}_h\|^2_{L^2(\Omega)}) \tag{32}$$

The first term in C (call it C_1) may be written using Fourier transforms and the second (call it C_2) will be estimated later.

$$C_1 = 4\pi h\int_{\mathbf{R}^2} |\widehat{\bar{m}}_{3,h}|^2(\eta)\hat{\Gamma}_h(|\eta|),$$

where $\widehat{m}_{3,h}$ is Fourier transform of $(\bar{m}_h \cdot e_3)$, $\Gamma_h(x)$ is defined in (8) and $\hat{\Gamma}_h$ is its Fourier transform, given by (7) . It is obvious that $\hat{\Gamma}_h(|\eta|) \geq 0$.

Using lemma 3 and the above estimates we conclude that for asymptotic regime $h \to 0$ the magnetostatic energy is

$$\begin{aligned}\frac{1}{h^2|\log h|}\int_{\mathbf{R}^3}|\nabla u_h|^2 &= \frac{1}{h|\log h|}\int_{\mathbf{R}^2}|\widehat{m}_{3,h}|^2(\eta)\hat{\Gamma}_h(|\eta|)\\ &+ \frac{1}{4\pi h^2|\log h|}\int_{\partial\omega}\int_{\partial\omega}\int_0^h\int_0^h \frac{(\bar{m}_h\cdot n)(x)(\bar{m}_h\cdot n)(y)}{\sqrt{|x-y|^2+(s-t)^2}}\\ &+ O\left(\frac{1}{|\log h|}\right)\left(\|\operatorname{div}_p \tilde{m}_h\|^2_{L^2(\Omega)} + \frac{1}{h^2}\|\frac{\partial \tilde{m}_h}{\partial z}\|^2_{L^2(\Omega)} + 1\right) \quad (33)\end{aligned}$$

As we see from the formula for the full micromagnetic energy (17) the last term in the above expression does not matter if $\frac{d^2}{h|\log h|}$ stays bounded away from 0. Since $\hat{\Gamma}_h(|\eta|) \to 1$ for $a.e\ \eta \in \mathbf{R}^2$ and $\hat{\Gamma}_h(|\eta|)$ is bounded, the second term asymptotically gives us the constraint $m_3 = 0$. So the only term we don't know anything about yet is C_2 (the second term of (29)). Let us prove the following lemma.

Lemma 4. *Assume $\partial\omega$ is sufficiently smooth, $\bar{m}_h \to \bar{m}$ weakly in $H^1(\omega)$ then we have*

$$\lim_{h\to 0}\frac{1}{h^2|\log h|}\int_{\partial\omega}\int_{\partial\omega}\int_0^h\int_0^h \frac{(\bar{m}_h\cdot n)(x)(\bar{m}_h\cdot n)(y)}{\sqrt{|x-y|^2+(s-t)^2}} = 2\int_{\partial\omega}(\bar{m}\cdot n)^2.$$

Proof. To simplify notation we set $f_h(x) = (\bar{m}_h \cdot n)(x)$, then rescaling in s and t variables we have

$$\begin{aligned}&\frac{1}{|\log h|}\int_0^1\int_0^1\int_{\partial\omega}\int_{\partial\omega}\frac{f_h(x)f_h(y)}{\sqrt{|x-y|^2+h^2(s-t)^2}}\\ &\quad= \frac{1}{|\log h|}\int_0^1\int_0^1\int_{\partial\omega} f_h^2(x)\int_{\partial\omega}\frac{1}{\sqrt{|x-y|^2+h^2(s-t)^2}}\\ &\qquad+ \frac{1}{|\log h|}\int_0^1\int_0^1\int_{\partial\omega}\int_{\partial\omega}\frac{(f_h(y)-f_h(x))f_h(x)}{\sqrt{|x-y|^2+h^2(s-t)^2}}. \quad (34)\end{aligned}$$

Let's call the integrals in the RHS of (34) I_1 and I_2, respectively and work on them separately. Using the fact that $|f_h| \leq 1$ we can estimate I_2 in the following way

$$I_2 \leq \frac{1}{|\log h|}\int_0^1\int_0^1\int_{\partial\omega}\int_{\partial\omega}\frac{|f_h(y)-f_h(x)|}{\sqrt{|x-y|^2+h^2(s-t)^2}}. \quad (35)$$

Now we use Hölder's inequality and the fact that

$$\|f_h\|_{H^{\frac{1}{2}}(\partial\omega)} \leq C(\|\bar{m}_h\|_{H^1(\omega)} + 1) \leq C$$

to obtain

$$I_2 \le \frac{1}{|\log h|}\left(\int_{\partial\omega}\int_{\partial\omega}\frac{|f_h(y)-f_h(x)|^2}{|x-y|^2}\right)^{\frac{1}{2}} \le \frac{C}{|\log h|}. \quad (36)$$

To deal with the integral I_1 we define the following kernel

$$K_h(x-y) = \int_0^1\int_0^1 \frac{1}{\sqrt{|x-y|^2+h^2(s-t)^2}}. \quad (37)$$

The integral I_1 now becomes

$$I_1 = \frac{1}{|\log h|}\int_0^1\int_0^1\int_{\partial\omega} f_h^2(x)\int_{\partial\omega}\frac{1}{\sqrt{|x-y|^2+h^2(s-t)^2}}$$
$$= \frac{1}{|\log h|}\int_{\partial\omega} f_h^2(x)\int_{\partial\omega} K_h(x-y). \quad (38)$$

It's not very difficult to verify that

$$\frac{1}{|\log h|}\int_{\partial\omega} K_h(x-y) \to 2 \quad (39)$$

uniformly in $x \in \partial\omega$, see [5].

Using this fact we obtain

$$I_1 \to 2\int_{\partial\omega}(\bar{m}\cdot n)^2.$$

Lemma is proved.

Now, using lemma 4, we may obtain the asymptotic expression for magnetostatic energy which we will use to show Γ-convergence of the micromagnetic energy.

4.3. Γ -convergence.

In this subsection we are going to prove the Γ- convergence stated as theorems 1 and 2 in Subsection 4.1. Let us recall the definition of the micromagnetic energy we are considering here (for simplicity of notation we drop all tildas)

$$E_h(m_h) = \frac{d^2}{h|\log h|}\int_\Omega\left(|\nabla' m_h|^2 + \frac{1}{h^2}(\frac{\partial m_h}{\partial z})^2\right) + \frac{1}{h^2|\log h|}\int_{\mathbf{R}^3}|\nabla u_h|^2.$$

Consider first the scaling $\frac{d^2}{h|\log h|} \to \infty$:

Proof of theorem 1. Let us prove the first statement. Using lemma 3 and (6) we know that

$$E_h(m_h) = \left(\frac{d^2}{h|\log h|} + o(1)\right) \int_\Omega \left(|\nabla' m_h|^2 + \frac{1}{h^2}(\frac{\partial m_h}{\partial z})^2\right) \\ + h \int_{\mathbf{R}^2} \frac{(\eta \cdot \widehat{\bar{m}'}_h)^2}{|\eta|^2} \left(1 - \hat{\Gamma}_h(|\eta|)\right) + h \int_{\mathbf{R}^2} \widehat{\bar{m}}^2_{3,h}(\eta) \hat{\Gamma}_h(|\eta|). \quad (40)$$

If energy is bounded $E_h(m_h) \leq C$, since $0 \leq \hat{\Gamma}_h \leq 1$ and $\frac{d^2}{h|\log h|} \to \infty$ we obtain the following inequalities

- $\|\nabla m_h\|_{L^2(\Omega)} \leq o(1)$;
- $\int_{\mathbf{R}^2} |\widehat{\bar{m}}_{3,h}|^2 \hat{\Gamma}_h(|\eta|) \leq Ch|\log h|$.

Therefore $m_h \to m$ and $m = const$, $|m| = 1$. It is easy to see that $\int_{\mathbf{R}^2} |\widehat{\bar{m}}_{3,h}|^2 \hat{\Gamma}_h(|\eta|) \to \int_{\mathbf{R}^2} |\widehat{\bar{m}}_3|^2$ and this implies $\bar{m}_3 = 0$. But since $m = const$ we have $m_3 = \bar{m}_3 = 0$. The first part of the theorem 1 is proved.

Let us show the second statement. Take any $m \in H^1(\Omega; S^2)$. Using (33) we see that for any $m_h \in H^1(\Omega; S^2)$

$$E_h(m_h) = \left(\frac{d^2}{h|\log h|} + o(1)\right) \int_\Omega \left(|\nabla' m_h|^2 + \frac{1}{h^2}(\frac{\partial m_h}{\partial z})^2\right) \\ + \frac{1}{4\pi} C_2 + \frac{1}{h|\log h|} \int_{\mathbf{R}^2} |\widehat{\bar{m}}_{3,h}|^2(\eta) \hat{\Gamma}_h(|\eta|). \quad (41)$$

Suppose $m_3 = 0$ and $m = const$. Then we may construct a sequence by taking $m_h = m$. Plugging this sequence in (41) and taking the limit as $h \to 0$ we have

$$E_h(m_h) = \frac{1}{4\pi} C_0 \to \frac{1}{2\pi} \int_{\partial\omega} |(m \cdot n)|^2.$$

Now for any sequence $m_h \to m$ in $H^1(\Omega; S^2)$ using lemma 4 and the first statement of the theorem we obviously have

$$\liminf E_h(m_h) \geq E_1(m).$$

Theorem 1 is proved.
Now we consider the scaling $\frac{d^2}{h|\log h|} \to \alpha$:

Proof of theorem 2. Proof of the first statement is analogous to that of theorem 1. Here we obtain that if $E_h(m_h) \leq C$ then $m_h \to m$ weakly in $H^1(\Omega)$ and m is independent of thickness variable z, $|m| = 1$, and $m_3 = 0$.

Let us show the second statement. Take any $m \in H^1(\Omega; S^2)$. Suppose $m_3 = 0$ and $m = m(x, y)$. Then we may construct a sequence by taking

$m_h = m$. Plugging this sequence in (41) and taking the limit as $h \to 0$ we have

$$E_h(m_h) = (\frac{d^2}{h|\log h|} + o(1)) \int_\Omega |\nabla' m_h|^2 + \frac{1}{4\pi} C_0$$
$$\to \alpha \int_\Omega |\nabla' m|^2 + \frac{1}{2\pi} \int_{\partial\omega} |(m \cdot n)|^2.$$

For any sequence $m_h \to m$ weakly in $H^1(\Omega; S^2)$ using lemma 4 and the first part of the theorem we have

$$\liminf E_h(m_h) \geq E_2(m).$$

Theorem 2 is proved.

5. Appendix A

Here we are going to prove some inequalities used in the paper.

Proof of lemma 1. We have

$$\int_\Omega \int_\Omega f(x)g(y)K(x-y) = \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} f(x)\chi_\Omega(x)g(y)\chi_\Omega(y)K(x-y)\chi_B(x-y),$$

where χ_B is a characteristic function of a large enough ball $B \subset \mathbf{R}^n$. If we denote $\tilde{f} = f(x)\chi_\Omega(x)$, $\tilde{g} = g(x)\chi_\Omega(x)$ $\tilde{K} = K(x)\chi_B(x)$ then

$$\left| \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \tilde{f}(x)\tilde{g}(y)\tilde{K}(x-y) \right|$$
$$= \left| \int_{\mathbf{R}^n} \tilde{f} \int_{\mathbf{R}^n} \tilde{K}(y)\tilde{g}(x-y) \right| \leq \int_{\mathbf{R}^n} |\tilde{K}(y)| \int_{\mathbf{R}^n} |\tilde{f}(x)||\tilde{g}(x-y)|$$
$$\leq \int_{\mathbf{R}^n} |\tilde{K}(y)| \left(\int_{\mathbf{R}^n} |\tilde{f}(x)|^2 \right)^{\frac{1}{2}} \left(\int_{\mathbf{R}^n} |\tilde{g}(x-y)|^2 \right)^{\frac{1}{2}}$$
$$= \|K\|_{L^1(B)} \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}. \quad (42)$$

Lemma is proved.

Proof of lemma 2. For any $y \in \omega$ we define the following function

$$h(y) = \int_\omega \frac{f(x)}{|x-y|}.$$

We obviously have the following estimate

$$\|h\|_{L^2(\omega)} \leq C\|f\|_{L^2(\omega)}.$$

Moreover it's easy to see that $\nabla^{\frac{1}{2}} h \in L^2(\omega)$ and may be estimated by

$$\|\nabla^{\frac{1}{2}} h\|_{L^2(\omega)} \le C\|f\|_{L^2(\omega)}.$$

Therefore $h \in H^{\frac{1}{2}}(\omega)$ and using Sobolev inclusion theorem we know that $H^{\frac{1}{2}}(\omega) \subset L^2(\partial\omega)$. Hence we have

$$\|h\|_{L^2(\partial\omega)} \le C\|h\|_{H^{\frac{1}{2}}(\omega)} \le C\|f\|_{L^2(\omega)}.$$

By definition of h, Hölder inequality and above arguments we have

$$\int_{\partial\omega}\int_{\omega} \frac{f(x)g(y)}{|x-y|} = \int_{\partial\omega} g(y)h(y) \le C\|f\|_{L^2(\omega)}\|g\|_{L^2(\partial\omega)}.$$

Lemma is proved.

6. Appendix B

For the reader's convenience, we derive the form of the magnetostatic energy in terms of Fourier transforms of m. The presentation here follows [9] but of course the calculation is much older, see [2].

We define n - dimensional Fourier transforms as

$$\hat{f}(\eta) = \int_{\mathbf{R}^n} f(x)\exp(-i2\pi x\cdot\eta)dx.$$

$$\int_{\mathbf{R}^3} |\nabla u|^2 = \int_{\mathbf{R}^3} |\widehat{\nabla u}|^2 = \int_{\mathbf{R}^3} \frac{|\eta\cdot\widehat{m\chi_{\Omega_h}}|^2}{|\eta|^2}.$$

Now

$$\widehat{m\chi_{\Omega_h}} = \int_{\Omega_h} m(x)\exp(-i2\pi x\cdot\eta) = \widehat{m\chi_\omega}\exp(-i\pi\eta_3 h)\frac{\sin(\eta_3\pi h)}{\pi\eta_3}.$$

Hence

$$\begin{aligned}|\widehat{m\chi_{\Omega_h}}\cdot\eta|^2 &= \frac{\sin^2(\eta_3\pi h)}{\pi^2\eta_3^2}|\widehat{m\chi_\omega}\cdot\eta|^2 \\ &= \frac{\sin^2(\eta_3\pi h)}{\pi^2\eta_3^2}(|\widehat{m'\chi_\omega}\cdot\eta'|^2 + |\widehat{m_3\chi_\omega}\eta_3|^2 + 2Re[(\widehat{m'\chi_\omega}\cdot\eta')\widehat{m_3\chi_\omega}\eta_3]),\end{aligned}$$

and we obtain

$$\begin{aligned}\int_{\mathbf{R}^3} |\nabla u|^2 = \int_{\mathbf{R}^2} |\widehat{m'\chi_\omega}\cdot\eta'|^2 \int_R \frac{\sin^2(\eta_3\pi h)}{\pi^2\eta_3^2(\eta_3^2+|\eta'|^2)} \\ + \int_{\mathbf{R}^2} |\widehat{m_3\chi_\omega}|^2 \int_R \frac{\sin^2(\pi\eta_3 h)}{\pi^2(\eta_3^2+|\eta'|^2)}.\end{aligned}$$

Now we calculate the following integrals

$$\int_R \frac{\sin^2(\eta_3 \pi h)}{\pi^2(\eta_3^2 + |\eta'|^2)} = h \frac{1 - \exp(-2\pi h|\eta'|)}{2\pi h|\eta'|} = h\hat{\Gamma}_h(\eta'),$$

$$\int_R \frac{\sin^2(\eta_3 \pi h)}{\pi^2 \eta_3^2(\eta_3^2 + |\eta'|^2)} = h \frac{1 - \hat{\Gamma}_h(\eta')}{|\eta'|^2}.$$

Finally we have

$$\int_{\mathbf{R}^3} |\nabla u|^2 = h \int_{\mathbf{R}^2} |\widehat{m'\chi_\omega} \cdot \eta'|^2 \frac{1 - \hat{\Gamma}_h(\eta')}{|\eta'|^2} + h \int_{\mathbf{R}^2} |\widehat{m_3\chi_\omega}|^2 \hat{\Gamma}_h(\eta').$$

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