Calculus of variations manuscript No.
(will be inserted by the editor)

Robert V. Kohn · Valeriy V. Slastikov

Geometrically Constrained Walls

Abstract. We address the effect of extreme geometry on a non-convex variational problem. The analysis is motivated by recent investigations of magnetic domain walls trapped by sharp thin necks. To capture the essential issues in the simplest possible setting, we focus on a scalar variational problem with a symmetric double well potential, whose spatial domain is a dumbell with a sharp neck. Our main results are (a) the existence of local minimizers representing geometrically constrained walls, and (b) an asymptotic characterization of the wall profile. Our analysis uses methods similar to Γ-convergence; in particular, the wall profile minimizes a certain "reduced problem" – the limit of the original problem, suitably rescaled near the neck. The structure of the wall depends critically on the choice of scaling, specifically the ratio between length and width of the neck.

1. Introduction

We study the nonconvex variational problem

$$\gamma \int_{\Omega} |\nabla u|^2 + \int_{\Omega} (u^2 - 1)^2, \tag{1}$$

focusing on the relation between the internal structure of the transition layers and the overall geometry of the domain. We assume that $\Omega \subset R^3$ is a dumbell-shaped domain like Figure 1 with a sharp, thin neck.

This problem models a *geometrically constrained wall* in a magnetic point contact. The extreme geometry of the domain acts like a singular

R. V. Kohn: Courant Institute, New York University, New York, NY 10012, USA
(e-mail: kohn@cims.nyu.edu)

V. V. Slastikov: Carnegie Mellon University, Pittsburgh, PA 15213, USA
(e-mail: slastiko@andrew.cmu.edu)

The research of the first author was supported in part by NSF grants DMS 0101439 and 0313744. RVK also acknowledges helpful discussions with Francois Murat, facilitated by funding from Laboratoire Jacques-Louis Lions at Université de Paris VI in Fall 2001.

The research of the second author was supported in part by NSF grant DMS-0405343. Address from September 2005: Department of Mathematics, University of Warwick, Coventry CV4 7AL, UK (e-mail: valeriy@maths.warwick.ac.uk). An early version of this work was part of VVS's PhD thesis at NYU.

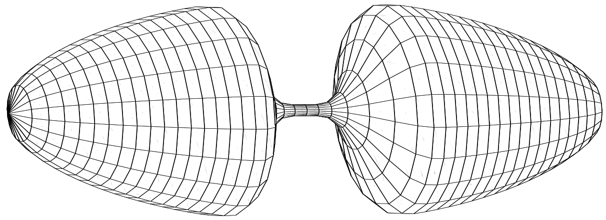

Fig. 1. *A dumbell-shaped domain with a small neck.*

perturbation of the functional. The problem was brought to our attention by recent work of P. Bruno [2]. He observed that when a ferromagnet has a thin neck, this will be a preferred location for a domain wall; moreover if the geometry of the neck varies rapidly enough, it can influence and even dominate the structure of the wall. The physical importance of this effect lies in its consequences for magnetoresistance, since one expects a strong correlation between exchange energy and electrical resistance [8].

To explain the basic phenomenon let us briefly summarize Bruno's discussion. He considered a symmetric planar ferromagnet as shown in Figure 2: the midplane is parallel to the $x-z$ plane, and the magnet occupies the 3D domain $|z| < S(x),\ |y| < h$. Suppose furthermore that the material is uniaxial, with $m = (0, 0, \pm 1)$ as the preferred magnetization directions. Finally, suppose the neck has trapped a wall, i.e. the magnetization is $m \approx (0, 0, -1)$ to the left of the neck and $m \approx (0, 0, 1)$ to the right of the neck. To understand the wall profile, Bruno assumed (as an ansatz) that the magnetization depends only on x and rotates in the $y - z$ plane

$$m = (0, \cos\theta(x), \sin\theta(x)), \tag{2}$$

and that magnetostatic interaction can be ignored in finding the wall profile. (These modeling hypotheses are familiar from the well-known analysis of a Bloch wall.) Minimizing the micromagnetic energy within this ansatz amounts to solving the one-dimensional calculus of variations problem

$$\min_{\theta(x)} \int [A\theta_x^2 + K\cos^2\theta(x)]S(x)dx$$

where $\theta \approx -\pi/2$ for $x \ll 0$ and $\theta \approx \pi/2$ for $x \gg 0$. Bruno's essential observation is that this problem has two independent length scales: the magnetic one $\ell_1 = \sqrt{A/K}$ and the geometric one $\ell_2 = \sqrt{S(0)/S''(0)}$. If these are well-separated the profile of the wall is governed by the smaller of the two; thus ℓ_1 governs and we obtain a Bloch wall if $S(0)/S''(0)$ is large enough, while ℓ_2 governs and the wall is much thinner than a Bloch wall if $S(0)/S''(0)$ is small enough.

Bruno's analysis is insightful, but its accuracy is limited by the simplicity of the ansatz (2). Subsequent work by Molyneux, Osipov and Ponizovskaya

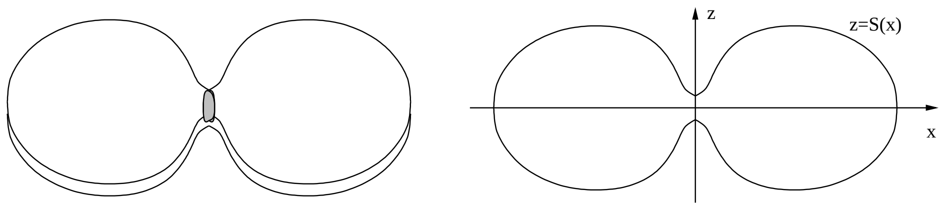

Fig. 2. *A film with a thin neck, capable of geometrically constraining a wall.*

[14] did better, by permitting m to vary in the $x-z$ plane. That paper simulates and analyzes the two-dimensional structure of a constrained wall in a thin film. It also considers point contacts, i.e. rotationally-symmetric three-dimensional problems like Figure 1. However the discussion by Molyneux et al. is, like Bruno's, largely formal – lacking rigorous mathematical content or justification.

There is a mathematical literature on problems of this type. The first results are due to Hale & Vegas [9], Jimbo [10,11]. More recent results include the papers by Cabib et al. [3], Rubinstein et al. [15], and the series of papers by Casado-Díaz et al [4–6]. All these authors address problems where the domain Ω_ϵ has a thin neck of length scale ϵ (in a suitable sense). Let us briefly describe some of the previous results.

Hale and Vegas consider a neck having a fixed length, shrinking only in the z direction. Using bifurcation theory they prove the existence of non-constant solutions for a particular nonlinear Poisson equation on a domain with a thin neck. The stability of these solutions depends on the stability of solutions of a certain limiting problem. This approach is rather technical, and difficult to apply in our case when we have the neck shrinking in both the x and radial directions. Moreover Hale and Vegas did not investigate the local structure of their solution within the neck.

Jimbo's articles use similar methods, but obtain additional information on the structure of the solution. His arguments work for necks in dumbell-shaped domains, having fixed length and shrinking in radial direction.

The analysis in the present paper is quite different from the work just summarized. It is in fact closer to the recent work by Casado-Díaz et al. on the thermal behavior of a notched beam [6]. They study a steady-state diffusion equation on the cylindrical domain

$$\left\{(x,y,z) \in \mathbf{R}^3 \ : \ -1 < x < 1, \sqrt{y^2+z^2} < \epsilon d_\epsilon(x)\right\}$$

when the diameter takes two values

$$d_\epsilon = \begin{cases} 1 & \text{for } |x| > t_\epsilon \\ r_\epsilon & \text{for } |x| < t_\epsilon, \end{cases}$$

examining the limiting behavior as ϵ, t_ϵ, and r_ϵ go to 0. Note that when t_ϵ and r_ϵ are both small the notch is in fact a sharp neck. The authors show that the solution is asymptotically one-dimensional, but the limit depends on how t_ϵ and r_ϵ scale with ϵ. Their method involves rescaling the solution near the neck, then applying compactness arguments to show convergence of the rescaled functions to the solution of an appropriate 1D problem. Our problem is rather different – the variational problem is not quadratic, and the limiting behavior is only one-dimensional for a "thin neck" – but our use of scaling is somewhat similar.

Our main results are

(a) existence of local minimizers that can be viewed as geometrically constrained walls, and
(b) identification of the associated wall profiles, asymptotically as the diameter and length of the wall tend to 0.

The wall profiles depend on the choice of scaling, specifically on the ratio between the two length scales

$$\delta = \text{diameter of the neck, and}$$
$$\epsilon = \text{length of the neck}$$

(see Section 2 for more careful definitions). There are three main cases:

- *a thin neck*, corresponding to $\delta/\epsilon \to 0$;
- *a normal neck*, corresponding to $\delta/\epsilon \to$ const; and
- *a thick neck*, corresponding to $\delta/\epsilon \to \infty$.

We'll show that for a thin neck, the wall is essentially one-dimensional and it is confined to the neck; in this case Bruno's ansatz (2) is asymptotically correct. For a normal or thick neck, however, the situation is different: the wall is *not* one-dimensional, and it is *not* confined to the neck. Rather, it spreads well into the regions on either side of the neck, as shown schematically in Figure 3.

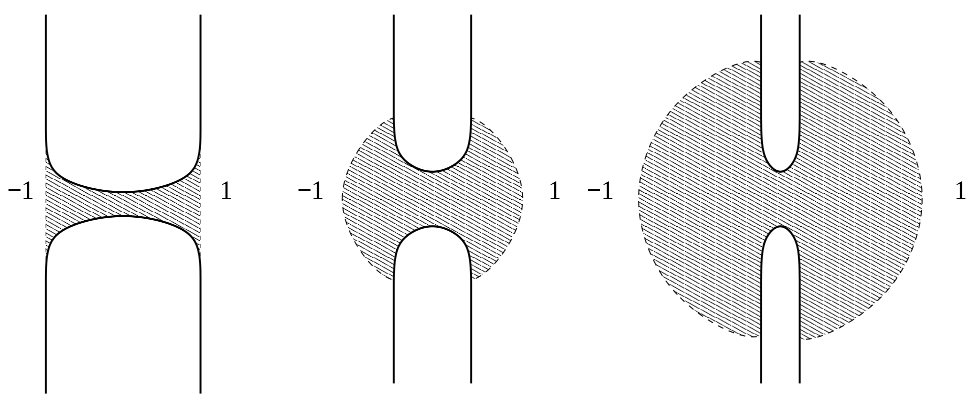

Fig. 3. *The region occupied by the wall, for thin (left), normal (center) and thick (right) necks.*

Our approach is entirely variational. To prove the existence of a geometrically constrained wall, we apply an idea from [12]. Briefly: we minimize the energy globally subject to a constraint that assures the desired behavior outside the neck; then we show that when δ and ϵ are small the constraint cannot be active. To characterize the asymptotic profile we show that our local minimizers, suitably rescaled, converge to a minimizer of an appropriate "asymptotic variational problem." Though motivated by the framework of Γ-convergence [1], our arguments are in fact self-contained and rather elementary.

We prove the existence of *at least one* geometrically constrained wall. We do not however attempt to classify or characterize *all* geometrically constrained walls. In the notation of Section 3: it is natural to ask whether the functional F_ϵ has other local minima besides the ones obtained variationally in Theorem 1. This question is presently open.

If other geometrically constrained walls do exist, it is natural to inquire about their profiles. Some of our results may be applicable; for example, the conclusion that in a thin neck the wall profile is one-dimensional requires little more than a suitable bound on the wall energy. But our characterization of the asymptotic profile (as the minimizer of an asymptotic variational problem) seems to make essential use of our variational scheme for identifying the local minimum.

2. Formulation of the problem

As we explained in the Introduction, both the 3D (Figure 1) and 2D (Figure 2) versions of our problem are interesting for applications. The two versions seem very similar, however our approach does not work well in 2D. Therefore we shall focus entirely on the 3D version, aside from Section 5 where we explain why the 2D setting is different.

Our goal is to understand the basic phenomenon, not to prove the most general possible theorem. Therefore we focus on a fairly simple class of solids of revolution about the x axis. We describe them by specifying their 2D sections in the x, z plane. Their geometry is determined by:

(a) a positive function $f : [-1, 1] \mapsto \mathbf{R}_+$, which determines (after scaling) the shape of the neck;
(b) a plane domain Ω^l, whose rotation about the x-axis is (up to translation) the part of Figure 1 to the left of the neck;
(c) a plane domain Ω^r, whose rotation about the x-axis is (up to translation) the part of Figure 1 to the right of the neck; and
(d) small parameters $\epsilon > 0$ and $\delta > 0$ which determine the scaling of the neck.

The associated domain is the union of three pieces,

$$\Omega_\epsilon = \Omega^l_\epsilon \cup R_\epsilon \cup \Omega^r_\epsilon, \tag{3}$$

namely the parts to the left and right of the neck

$$\Omega^l_\epsilon = \Omega^l - (\epsilon, 0, 0) \quad \text{and} \quad \Omega^r_\epsilon = \Omega^r + (\epsilon, 0, 0) \tag{4}$$

and the neck itself

$$R_\epsilon = \{(x,y,z) \,:\, \sqrt{y^2+z^2} < \delta f(x/\epsilon)\} \tag{5}$$

(see Figure 4). We write Ω_ϵ not $\Omega_{\epsilon,\delta}$ for notational simplicity, and because the asymptotic behavior depends on the limiting value of δ/ϵ – so it is natural to take the viewpoint that δ depends on ϵ. By a harmless abuse of notation, we shall not distinguish notationally between a 2D domain (symmetric about the x axis) and the associated 3D solid of revolution.

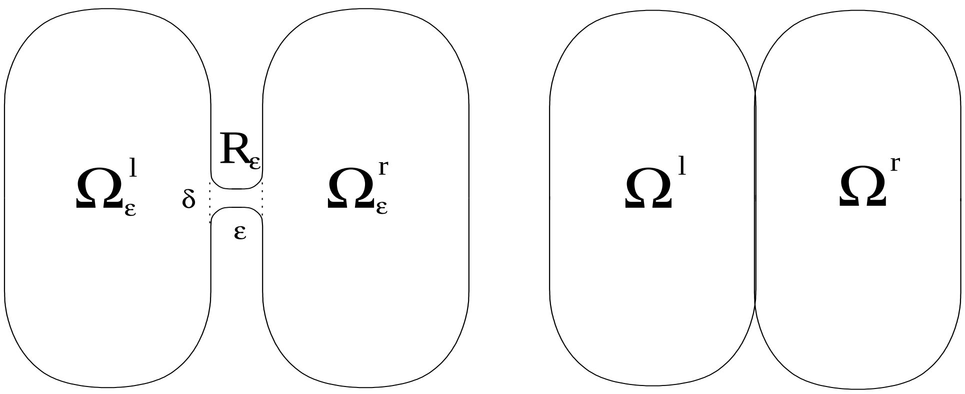

Fig. 4. *Sections of Ω_ϵ (left) and the limiting domain Ω_0 (right).*

We make some assumptions that, while perhaps implicit in the figure, have not yet been made explicit. The domain Ω^l lies in the left half-plane $x < 0$, and its right hand boundary is flat (a line segment) in a neighborhood of the x axis. Similarly, Ω^r lies in the right half-plane $x > 0$, and its left hand boundary is flat near the x axis. We always assume that the neck meets only this flat part (this is a smallness condition on δ). Further, we suppose for normalization that

$$f(-1) \le 1, \quad f(1) \le 1. \tag{6}$$

In our figures Ω_ϵ is a C^1 domain, i.e. $f'(-1) = -\infty$ and $f'(1) = \infty$, however our analysis does not require this. We assume only that Ω_ϵ is a Lipschitz domain (so we can use standard Sobolev-type inequalities). Thus our analysis permits f to be constant (a cylindrical neck), Lipschitz continous, or even piecewise Lipschitz with finitely many discontinuities.

Note that while the neck is both short (length 2ϵ) and thin (diameter of order δ), the left and right sides of the domain remain of order one. Therefore the limiting domain as $\epsilon \to 0$ and $\delta \to 0$ is the *disjoint union*

$$\Omega_0 = \Omega^l \cup \Omega^r. \tag{7}$$

An H^1 function on Ω_0 can be discontinuous at the surface where $\partial\Omega^l$ meets $\partial\Omega^r$, since this surface represents a “crack”.

As announced in the Introduction, our idea is to view a geometrically constrained wall as a local minimizer of a scalar double-well variational problem

$$F_\epsilon(u) = \gamma_\epsilon \int_{\Omega_\epsilon} |\nabla u|^2 + \int_{\Omega_\epsilon} (u^2-1)^2 \tag{8}$$

defined on the rotationally-symmetric 3D domain Ω_ϵ. The coefficient in front of the gradient term is allowed to depend on ϵ, but it cannot be too small. In proving the existence of geometrically constrained walls (Theorem 1) we will assume

$$\gamma_\epsilon \geq C\delta^2 \tag{9}$$

with $C > 0$ independent of ϵ and δ. In characterizing the profiles of our geometrically constrained walls (Theorems 2 - 7) we will assume the stronger condition

$$\gamma_\epsilon \geq C \max\{\epsilon^2, \delta^2\}. \tag{10}$$

This lower bound assures that our scaled variational problems are not of Modica-Mortola type, i.e. that in our scaled variables the wall profile does not degenerate to a sharp interface. (When $\gamma_\epsilon / \max\{\epsilon^2, \delta^2\} \to 0$ our asymptotic variational problems are indeed of Modica-Mortola type; this is clear from (35), (40), and (46).)

We also require that γ_ϵ not be too large, namely that

$$\gamma_\epsilon(\delta + \epsilon) \to 0. \tag{11}$$

This condition is used in the proof of existence, Theorem 1. Notice that (11) permits $\gamma_\epsilon \to \infty$. Such behavior is consistent with the existence of a wall because the size of the neck tends to 0. We're not sure (11) is optimal – perhaps it can be weakened – but it's clear that *some* upper bound is required. Indeed, when $\gamma_\epsilon \to \infty$ with ϵ, δ held fixed, finiteness of the energy requires u to be constant, precluding the existence of a wall.

3. Existence of geometrically constrained walls

This section proves that when ϵ and δ are sufficiently small, the functional F_ϵ defined by (8) has a local minimizer that's approximately -1 to the left of the neck and $+1$ to the right of the neck.

Our proof is self-contained, but the strategy is taken from [12]. It uses the simple observation that for any sequences $\epsilon_j, \delta_j \to 0$ and $\gamma_j \to \gamma_0$, the functionals F_{ϵ_j} converge to

$$F_0(u) = \gamma_0 \int_{\Omega_0} |\nabla u|^2 + \int_{\Omega_0} (u^2-1)^2 \tag{12}$$

where Ω_0 is the disjoint union of Ω^l and Ω^r. We'll show that

$$u_0 = \begin{cases} -1 \text{ in } \Omega^l \\ +1 \text{ in } \Omega^r \end{cases} \tag{13}$$

is, in a suitable sense, an isolated local minimizer of F_0. It follows via the method of [12] that F_j has a local minimizer near u_0 for all sufficiently large j. Examining the proof, we'll see that the conclusion doesn't depend on the choice of a sequence or the hypothesis that $\{\gamma_j\}$ converges.

In the definition (12) of F_0 we permit $\gamma_0 = 0$ or $\gamma_0 = \infty$. In the former case the gradient term is absent; in the latter case F_0 is infinite unless u is constant, and $F_0 = (u^2-1)^2|\Omega|$ when u is constant.

To get started, let's explain the sense in which u_0 is an isolated local minimizer of F_0.

Definition 1. *A function $u \in H^1(\Omega_0)$ is an isolated L^2-local minimizer of F_0 if, for some $d > 0$, $F_0(u) < F_0(v)$ for all $v \in H^1(\Omega_0)$ such that $0 < \|v-u\|_{L^2(\Omega_0)} < d$.*

This definition might seem a bit strange when $\gamma_0 = 0$ since we stipulate that u and v be in $H^1(\Omega_0)$ though F_0 is finite for any function in L^4. But when we make use of this definition, in the proof of Theorem 1, the functions in question will be limits of (constrained) minimizers. Therefore we will have estimates beyond simply knowing that $F_0 < \infty$. In particular we will have an H^1 bound.

Lemma 1. *The function u_0 is an isolated L^2-local minimizer of the functional F_0 in the sense of Definition 1.*

Proof: Since $F_0(u_0) = 0$, we have only to show (for some d) that $F_0(v) > 0$ for all $v \in H^1(\Omega_0)$ such that $0 < \|v-u_0\|_{L^2(\Omega_0)} < d$. But $F_0(v) > 0$ for all $v \in H^1(\Omega_0)$ unless $v = \pm 1$. The L^2 distance to u_0 is smallest when v agrees with u_0 on one side and has the opposite sign on the other. So the assertion is valid whenever $d < d_{\max}$ where

$$d_{\max} = \min\left\{|\Omega^l|^{1/2}, |\Omega^r|^{1/2}\right\} \tag{14}$$

□

Our argument proving the existence of geometrically constrained walls is variational, so we need some good test functions. The following Lemma is based on an ansatz for the wall profile that's more or less the opposite of Bruno's (2). Informally: our test function ξ_ϵ vanishes in the neck; to the left and right of the neck it varies radially (in coordinates centered at $(-\epsilon, 0, 0)$ and $(\epsilon, 0, 0)$ respectively); beyond radius R it equals ± 1 (see Figure 5). A similar construction was used (for a similar purpose) in [7]. When $\delta \ll \epsilon$ or $\delta \sim \epsilon$ we could alternatively have used (2); the test function ξ_ϵ has the advantage of being useful even when $\delta \gg \epsilon$.

Lemma 2. *For any $\epsilon > 0$ and $\delta > 0$ consider the 3D axially-symmetric domain Ω_ϵ defined by (3), and the functional F_ϵ defined by (8). Let the test*

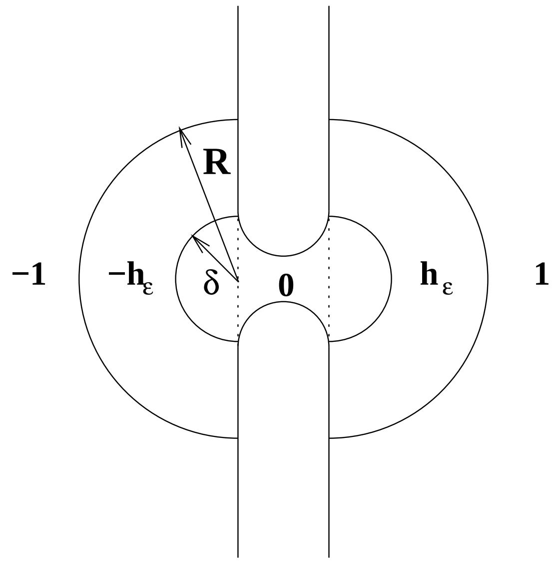

Fig. 5. *The test function ξ_ϵ.*

function $\xi_\epsilon : \Omega_\epsilon \mapsto \mathbf{R}$ be defined by (see Figure 5)

$$\xi_\epsilon(x,y,z) = \begin{cases} -1 & \text{when } |(x+\epsilon,y,z)| \geq R \text{ and } x < -\epsilon \\ -h(x+\epsilon,y,z) & \text{when } |(x+\epsilon,y,z)| \leq R \text{ and } x < -\epsilon \\ 0 & \text{when } -\epsilon \leq x \leq \epsilon,\ \sqrt{y^2+z^2} < \delta f(x/\epsilon) \\ h(x-\epsilon,y,z) & \text{when } |(x-\epsilon,y,z)| \leq R \text{ and } x > \epsilon \\ 1 & \text{when } |(x-\epsilon,y,z)| \geq R \text{ and } x > \epsilon, \end{cases}$$

where $h : \mathbf{R}^3 \to \mathbf{R}$ solves Laplace's equation on $B_R \backslash B_\delta$ with $h = 1$ beyond R and $h = 0$ within radius δ:

$$h = \begin{cases} 1 & \text{for } r \geq R \\ \frac{1/r - 1/\delta}{1/R - 1/\delta} & \text{for } \delta \leq r \leq R \\ 0 & \text{for } r \leq \delta \end{cases}$$

Taking $R = 2\delta$, and assuming that γ_ϵ satisfies (9), we have the estimate

$$\frac{1}{\gamma_\epsilon} F(\xi_\epsilon) \leq C(\delta + \epsilon). \tag{15}$$

The constant in (15) depends only on the L^∞ norm of the neck profile f and the constant in (9); in particular it does not require any relation between ϵ and δ.

Proof: The function ξ_ϵ is continuous, since by (6) the regions where it is nonzero do not meet the neck. Therefore it is in H^1 and

$$\int_{\Omega_\epsilon} |\nabla \xi_\epsilon|^2 = \int_{B_R \backslash B_\delta} |\nabla h|^2.$$

Since $|\nabla h| = \frac{R\delta}{R-\delta} r^{-2}$ this gives

$$\int_{\Omega_\epsilon} |\nabla \xi_\epsilon|^2 \leq 4\pi \left(\frac{R\delta}{R-\delta}\right)^2 \int_\delta^R r^{-2}\,dr = 4\pi \frac{R\delta}{R-\delta}. \tag{16}$$

As for the potential term, we have

$$\begin{aligned}\int_{\Omega_\epsilon} (\xi_\epsilon^2 - 1)^2 \leq \int_{B_R \setminus B_\delta} (h^2-1)^2 + \int_{B_\delta \cup R_\epsilon} 1 \ &\leq \ \text{meas}(B_R) + \text{meas}(R_\epsilon) \\ &\leq \frac{4}{3}\pi R^3 + 2\pi\epsilon\delta^2 \|f\|^2_{L^\infty}. \end{aligned} \tag{17}$$

Combining (16) and (17) and making the choice $R = 2\delta$, we obtain an inequality of form

$$\frac{1}{\gamma_\epsilon} F(\xi_\epsilon) \leq C\left(\delta + \frac{\delta^3}{\gamma_\epsilon} + \frac{\delta^2\epsilon}{\gamma_\epsilon}\right)$$

where C depends only on $\|f\|_\infty$. Combining this with (9) leads immediately to the desired bound (15). □

We are ready to prove the existence of geometrically constrained walls. Recall the function u_0 defined by (13). Let $u_{0,\epsilon}$ be the analogous function defined on Ω_ϵ:

$$u_{0,\epsilon} = \begin{cases} -1 \text{ in } \Omega_\epsilon^l \\ \ \ 0 \text{ in } R_\epsilon \\ +1 \text{ in } \Omega_\epsilon^r. \end{cases} \tag{18}$$

Our geometrically constrained walls are L^2-local minimizers of F_ϵ, obtained by minimizing the functional *globally* on the set

$$B_\epsilon = \{u \in H^1(\Omega_\epsilon) \text{ such that } \|u - u_{0,\epsilon}\|_{L^2(\Omega_\epsilon)} \leq d\}. \tag{19}$$

The value of d in the definition of B_ϵ can be any number such that $0 < d < d_{\max}$ where $d_{\max}$ is defined by (14).

Theorem 1. *Let u_ϵ achieve*

$$\min_{u \in B_\epsilon} F_\epsilon(u). \tag{20}$$

Then

(a) there is a constant c_0 such that for all $\epsilon, \delta < c_0$ the function u_ϵ is an L^2-local minimizer of F_ϵ; and
(b) as $\epsilon, \delta \to 0$ we have $\|u_\epsilon - u_{0,\epsilon}\|_{L^2(\Omega_\epsilon)} \to 0$.

The value of c_0 depends only on $\|f\|_{L^\infty}$ and the bounds implicit in (9) and (11).

Proof: We begin by observing that

$$\max |u_\epsilon| \leq 1. \tag{21}$$

Indeed: if $|u_\epsilon|$ were to exceed 1 we could consider the test function obtained by truncating it above by 1 and below by -1. This test function would have smaller energy, and would still lie in B_ϵ. So u_ϵ would not have achieved the minimum of F_ϵ on B_ϵ.

To prove assertion (a) we argue by contradiction. Notice that if u_ϵ lies in the interior of B_ϵ then it is indeed an L^2-local minimizer. So if (a) is false there must be a sequence $\epsilon_k, \delta_k \to 0$ for which the associated u_{ϵ_k} lies at the boundary of B_ϵ, in other words

$$\|u_{\epsilon_k} - u_{0,\epsilon}\|_{L^2(\Omega_\epsilon)} = d.$$

Our test function ξ_ϵ lies in B_ϵ when δ is sufficiently small. Restricting attention to such δ, Lemma 2 gives

$$\frac{1}{\gamma_{\epsilon_k}} F_{\epsilon_k}(u_{\epsilon_k}) \leq \frac{1}{\gamma_{\epsilon_k}} F_{\epsilon_k}(\xi_{\epsilon_k}) \leq C(\epsilon_k + \delta_k),$$

whence

$$\int_{\Omega_\epsilon^l \cup \Omega_\epsilon^r} |\nabla u_{\epsilon_k}|^2 \to 0.$$

The domain of definition of u_{ϵ_k} varies with k; to pass to a limit it is convenient to consider its "translated restriction" to the left and right sides of the domain:

$$u_k^l(x,y,z) = u_{\epsilon_k}(x - \epsilon, y, z) \quad \text{for } (x,y,z) \in \Omega^l \text{ and}$$
$$u_k^r(x,y,z) = u_{\epsilon_k}(x + \epsilon, y, z) \quad \text{for } (x,y,z) \in \Omega^r.$$

The domains of u_k^l and u_k^r are independent of k, and the preceding estimate says

$$\int_{\Omega^l} |\nabla u_k^l|^2 \to 0, \qquad \int_{\Omega^r} |\nabla u_k^r|^2 \to 0.$$

It follows (passing to a subsequence if necessary) that each sequence converges to a limit:

$$\|u_k^l - u_*^l\|_{H^1(\Omega^l)} \to 0 \quad \text{and} \quad \|u_k^r - u_*^r\|_{H^1(\Omega^r)} \to 0 \tag{22}$$

and the limits u_*^l, u_*^r are constant. We view them as defining a function u_* on the limiting domain $\Omega_0 = \Omega^l \cup \Omega^r$.

Consider the limit u_*. We claim that $F_0(u_*) = 0$, where F_0 is the limiting energy, defined by (12). (We assume – without loss of generality, passing to a further subsequence if necessary – that $\gamma_{\epsilon_k} \to \gamma_0$ for some $0 \leq \gamma_0 \leq \infty$ so the limiting energy is well-defined.) Indeed, if $\gamma_0 < \infty$ we have

$$\begin{aligned} F_0(u_*) &= \int_{\Omega^l} \gamma_0 |\nabla u_*|^2 + (u_*^2 - 1)^2 + \int_{\Omega^r} \gamma_0 |\nabla u_*|^2 + (u_*^2 - 1)^2 \\ &= \lim_k \int_{\Omega^l} \gamma_{\epsilon_k} |\nabla u_k^l|^2 + ((u_k^l)^2 - 1)^2 + \lim_k \int_{\Omega^r} \gamma_{\epsilon_k} |\nabla u_k^r|^2 + ((u_k^r)^2 - 1)^2 \end{aligned}$$

since (22) implies strong convergence in L^4. It follows that

$$F_0(u_*) \leq \liminf_k F_{\epsilon_k}(u_{\epsilon_k}) \leq \liminf_k F_{\epsilon_k}(\xi_{\epsilon_k}) = 0$$

using (11) and (15). When $\gamma_0 = \infty$ the argument is similar.

Now recall that $\|u_{\epsilon_k} - u_{0,\epsilon}\|_{L^2(\Omega_\epsilon)} = d$. We claim that this implies

$$\|u_* - u_0\|_{L^2(\Omega_0)} = d$$

where u_0 is defined by (13). Indeed, the contribution of the neck to the L^2 norm is negligible since the volume of the neck tends to 0 and the functions u_{ϵ_k}, $u_{0,\epsilon}$ are uniformly bounded by ± 1. So

$$\lim_k \int_{\Omega_{\epsilon_k}} |u_{\epsilon_k} - u_{0,\epsilon}|^2 = \lim_k \int_{\Omega^l} |u^l_k + 1|^2 + \lim_k \int_{\Omega^r} |u^r_k - 1|^2 = \int_{\Omega_0} |u_* - u_0|^2.$$

We have reached a contradiction, since by Lemma 1 u_0 is an isolated local minimizer of F_0. Thus u_ϵ lies in the interior of B_ϵ for all sufficiently small ϵ and δ, and as a consequence it is an L^2-local minimizer of F_ϵ.

Turning to assertion (b), we argue again by contradiction. If the assertion is false then there is a sequence $\epsilon_k, \delta_k \to 0$ for which the associated u_{ϵ_k} satisfies

$$c \leq \|u_{\epsilon_k} - u_{0,\epsilon}\|_{L^2(\Omega_\epsilon)} \leq d$$

with $c > 0$. Arguing as in the proof of (a), there is a limit u_*, which satisfies

$$F_0(u_*) = 0 \quad \text{and} \quad c \leq \|u_* - u_0\|_{L^2(\Omega_0)} \leq d.$$

But this contradicts Lemma 1. So assertion (b) is valid. □

4. The wall profile

We view the local minimizers provided by Theorem 1 as geometrically constrained walls. This section examines their behavior near the neck, i.e. the profiles of these walls. The answer depends on the relationship between δ and ϵ as they tend to 0. When $\delta/\epsilon \to 0$ (a *thin neck*) the profile solves a 1D variational problem and is confined to the neck. When δ/ϵ has a finite, nonzero limit (a *normal neck*) the profile solves a 3D variational problem and the wall extends well beyond the neck. When $\delta/\epsilon \to \infty$ (a *thick neck*) the profile still solves a 3D variational problem, but the shape of the neck becomes irrelevant.

We shall address the three regimes separately. The analysis follows the same overall pattern in each case: we (a) choose an appropriate scaling; (b) use a suitable test function to bound the energy of the (rescaled) profile; then (c) apply variational arguments to characterize the optimal profile.

4.1. A thin neck

We assume in this subsection that $\delta = \delta(\epsilon)$ with $\delta/\epsilon \to 0$. The appropriate scaling is then

$$x \to x/\epsilon, \quad y \to y/\delta, \quad z \to z/\delta$$

and our hypothesis (10) becomes

$$\gamma_\epsilon \geq C\epsilon^2.$$

We shall work with the rescaled functional $\mathcal{F}_\epsilon$, defined by

$$\mathcal{F}_\epsilon(v) = \frac{\epsilon}{\delta^2\gamma_\epsilon} F_\epsilon(u) = \int_{\omega_\epsilon} \left[v_x^2 + \left(\frac{\epsilon}{\delta}\right)^2 (v_y^2 + v_z^2) \right] + \frac{\epsilon^2}{\gamma_\epsilon} \int_{\omega_\epsilon} (v^2 - 1)^2 \quad (23)$$

where

$$v(x, y, z) = u(x\epsilon, y\delta, z\delta) \quad (24)$$

and ω_ϵ is the image of Ω_ϵ under this change of variables. Let

$$\omega_0 = \lim_{\epsilon \to 0} \omega_\epsilon = \omega^l \cup \rho \cup \omega^r$$

be the limiting rescaled domain. It consists of two half-spaces

$$\omega^l = \{x < -1\}, \quad \omega^r = \{x > 1\}$$

connected by the rescaled neck

$$\rho = \{\sqrt{y^2 + z^2} < f(x), \ |x| \leq 1\}$$

(see Figure 6).

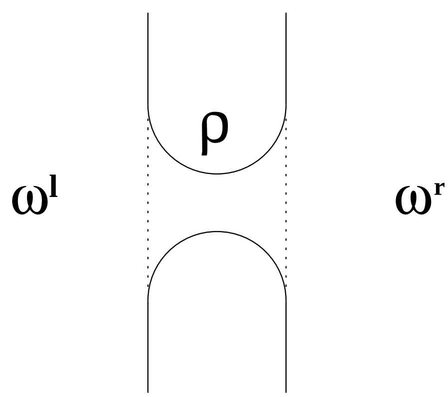

Fig. 6. *The limiting rescaled domain ω_0.*

Let u_ϵ be the local minimizer of F_ϵ supplied by Theorem 1, i.e. u_ϵ is any global minimizer of (20) (we do not assert uniqueness). Our goal is to characterize the behavior of v_ϵ, defined as the image of u_ϵ under the scaling (24). The situation is easy to understand heuristically. Since $\delta/\epsilon \to 0$ the terms involving v_y and v_z in (23) can be viewed as penalizations favoring $v_y = v_z = 0$. Therefore v_ϵ is asymptotically a function of x alone. If in addition $\epsilon^2/\gamma_\epsilon \to 0$ then the potential term is asymptotically irrelevant and the asymptotic wall profile minimizes the integral of v_x^2.

Our task is to make the preceding argument rigorous.

Lemma 3. *For all sufficiently small ϵ and δ the rescaled energy satisfies:*

$$\mathcal{F}_\epsilon(v_\epsilon) \le C\left(1 + \frac{\epsilon^2}{\gamma_\epsilon}\right)\|f\|^2_{L^\infty}. \tag{25}$$

In particular, if $\epsilon^2/\gamma_\epsilon$ is uniformly bounded then so is the rescaled energy $\mathcal{F}_\epsilon(v_\epsilon)$.

Proof: Consider the one-dimensional test function

$$\eta_\epsilon = \begin{cases} -1 \text{ in } \Omega^l_\epsilon \\ x/\epsilon \text{ in } R_\epsilon \\ +1 \text{ in } \Omega^r_\epsilon. \end{cases}$$

When ϵ is small so is the volume of the neck:

$$|R_\epsilon| \le 2\pi\epsilon\delta^2\|f\|^2_{L^\infty}.$$

Therefore η_ϵ is an admissible test function for the variational problem (20) that defines u_ϵ, i.e. $\eta_\epsilon \in B_\epsilon$. It follows that

$$F_\epsilon(u_\epsilon) \le F_\epsilon(\eta_\epsilon) \le \left(\frac{\gamma_\epsilon}{\epsilon^2} + 1\right)|R_\epsilon|,$$

whence

$$\mathcal{F}_\epsilon(v_\epsilon) = \frac{\epsilon}{\delta^2\gamma_\epsilon}F_\epsilon(u_\epsilon) \le C\left(1 + \frac{\epsilon^2}{\gamma_\epsilon}\right)\|f\|^2_{L^\infty}$$

as asserted. □

Here is our main result for the case $\epsilon^2/\gamma_\epsilon \to 0$, when the potential term is asymptotically irrelevant.

Theorem 2. *Suppose ϵ and $\delta = \delta(\epsilon) \to 0$ with*

$$\frac{\delta}{\epsilon} \to 0 \quad \textit{and} \quad \frac{\epsilon^2}{\gamma_\epsilon} \to 0.$$

Then the rescaled profiles v_ϵ of our local minimizers u_ϵ converge to the minimizer of

$$\min_{v \in \mathcal{A}} \int_\omega |v_x|^2 \tag{26}$$

where the admissible set is restricted to one-dimensional profiles that are constant outside the neck:

$$\mathcal{A} = \{v \in H^1_{loc}(\omega) \,:\, v_y = v_z = 0,\ v = -1 \text{ in } \omega^l,\ v = 1 \text{ in } \omega^r\}. \tag{27}$$

The convergence is strong in H^1 on compact subsets of ω.

Proof: The limiting variational problem (26) is extremely simple: it amounts to the one-dimensional calculus of variations problem

$$\min_{\substack{v(-1)=-1\\ v(1)=1}} \int_{-1}^{1} v_x^2 f(x)^2 \, dx. \tag{28}$$

The functional is strictly convex, so its minimizer is unique. Our expression (26) may seem overly complicated in this simple setting; however it facilitates comparison with the other cases (normal and thick necks), c.f. (39) and (44).

STEP 1. We claim that after passing to a sequence if necessary, $v_\epsilon \to v$ weakly in $H^1_{\text{loc}}(\omega)$, with

$$\int_\omega |\nabla v|^2 \leq \liminf_\epsilon \int_{\omega_\epsilon} |\nabla v_\epsilon|^2$$

and

$$v_y = v_z = 0, \text{ i.e. } v \text{ is a function of } x \text{ alone.}$$

(We continue to write v_ϵ not v_{ϵ_j} for notational simplicity.) Indeed, by Lemma 3 we have a uniform bound on $\mathcal{F}_\epsilon(v_\epsilon)$, and this implies a uniform bound on $\int_{\omega_\epsilon} |\nabla v_\epsilon|^2$. We also know from (21) that $|v_\epsilon| \leq 1$. Therefore weak limits exist on every bounded $\omega_b \subset \omega$ whose closure is contained in ω. Letting $\omega_b \uparrow \omega$ and using a standard diagonal construction, we conclude existence of a sequence converging to a limit v weakly in $H^1_{\text{loc}}(\omega)$. For each ω_b we have

$$\int_{\omega_b} |\nabla v_\epsilon|^2 \leq \liminf_\epsilon \int_{\omega_\epsilon} |\nabla v_\epsilon|^2$$

since $\omega_b \subset \omega_\epsilon$ when ϵ is sufficiently small. It follows by lower-semicontinuity that

$$\int_{\omega_b} |\nabla v|^2 \leq \liminf_\epsilon \int_{\omega_\epsilon} |\nabla v_\epsilon|^2$$

and applying the monotone convergence theorem as $\omega_b \uparrow \omega$ we conclude that

$$\int_\omega |\nabla v|^2 \leq \liminf_\epsilon \int_{\omega_\epsilon} |\nabla v_\epsilon|^2. \tag{29}$$

Since $\mathcal{F}_\epsilon(v_\epsilon)$ is uniformly bounded we have

$$\int_{\omega_\epsilon} v_y^2 + v_z^2 \leq C \left(\frac{\delta}{\epsilon}\right)^2 \to 0.$$

Arguing as for (29) we conclude that

$$\int_\omega v_y^2 + v_z^2 = 0,$$

so v is independent of y and z.

STEP 2. We claim that $v = -1$ in ω^l and $v = 1$ in ω^r. Focusing on ω^l, we shall use the scale-invariant Poincaré-type inequality

$$\left(\int_{\Omega_\epsilon^l} |u - \overline{u}|^6 \right)^{1/6} \leq C \left(\int_{\Omega_\epsilon^l} |\nabla u|^2 \right)^{1/2}, \tag{30}$$

where $\overline{u}$ is the average of u on Ω_ϵ^l. The constant is indendent of ϵ, because as ϵ varies the domains Ω_ϵ^l are identical up to translation and scaling. Applying (30) to u_ϵ then changing variables by (24) we have

$$\epsilon\delta^2 \int_{\omega_\epsilon^l} |v_\epsilon - \overline{v_\epsilon}|^6 \leq C \left(\epsilon\delta^2 \int_{\omega_\epsilon^l} \epsilon^{-2} v_{\epsilon,x}^2 + \delta^{-2}(v_{\epsilon,y}^2 + v_{\epsilon,z}^2) \right)^3,$$

whence

$$\begin{aligned} \left(\int_{\omega_\epsilon^l} |v_\epsilon - \overline{v_\epsilon}|^6 \right)^{1/3} &\leq C \left(\frac{\delta}{\epsilon} \right)^{4/3} \left[\int_{\omega_\epsilon^l} v_{\epsilon,x}^2 + \left(\frac{\epsilon}{\delta} \right)^2 (v_{\epsilon,y}^2 + v_{\epsilon,z}^2) \right] \\ &\leq C \left(\frac{\delta}{\epsilon} \right)^{4/3} \mathcal{F}_\epsilon(v_\epsilon) \end{aligned} \tag{31}$$

which tends to 0 since $\delta/\epsilon \to 0$ and $\mathcal{F}_\epsilon(v_\epsilon)$ is uniformly bounded.

Now recall from part (b) of Theorem 1 that $\|u_\epsilon + 1\|_{L^2(\Omega_\epsilon^l)} \to 0$. It follows (since the volume of Ω_ϵ^l is independent of ϵ) that the average of u_ϵ on Ω_ϵ^l tends to -1. In rescaled variables this says $\overline{v_\epsilon} \to -1$. Combining this result with (31) we conclude that for any compact $\omega_b \subset \omega^l$ we have

$$\int_{\omega_b} |v_\epsilon + 1|^6 \to 0.$$

We conclude (taking $\epsilon \to 0$ then $\omega_b \uparrow \omega^l$) that $v = -1$ on ω^l. The proof that $v = 1$ on ω^r is of course identical.

STEP 3. We have shown that v is in the admissible set $\mathcal{A}$ defined by (27). Now let's show that it minimizes the asymptotic functional (26), i.e. that

$$\int_\omega v_x^2 \leq \int_\omega \phi_x^2 \quad \text{for any } \phi \in \mathcal{A}. \tag{32}$$

Consider any $\phi \in \mathcal{A}$. We may suppose without loss of generality that

$$|\phi| \leq 1 \tag{33}$$

since truncating ϕ above by 1 and below by -1 maintains admissibility and decreases the value of $\int \phi_x^2$. We define a test function for F_ϵ by using ϕ for the wall profile:

$$\psi_\epsilon = \begin{cases} -1 & \text{in } \Omega_\epsilon^l \\ \phi(x/\epsilon) & \text{in } R_\epsilon \\ +1 & \text{in } \Omega_\epsilon^r. \end{cases}$$

The function ψ_ϵ is admissible for (20) for sufficiently small ϵ and δ, since

$$\int_{\Omega_\epsilon} |\psi_\epsilon - u_{0,\epsilon}|^2 = \epsilon\delta^2 \int_\rho \phi^2.$$

where ρ is the rescaled neck region (see Figure 6). We conclude from the definition (20) of u_ϵ that $F(u_\epsilon) \le F(\psi_\epsilon)$. In the rescaled variables this says

$$\mathcal{F}_\epsilon(v_\epsilon) \le \mathcal{F}_\epsilon(\phi_\epsilon) \tag{34}$$

where

$$\phi_\epsilon = \begin{cases} -1 & \text{in } \omega_\epsilon^l \\ \phi(x) & \text{in } \rho \\ +1 & \text{in } \omega_\epsilon^r. \end{cases}$$

Now,

$$\mathcal{F}_\epsilon(\phi_\epsilon) = \int_\rho \phi_x^2 + \frac{\epsilon^2}{\gamma_\epsilon}\int_\rho (\phi^2-1)^2 \to \int_\omega \phi_x^2$$

since $\epsilon^2/\gamma_\epsilon \to 0$. Combining this with (34) and (29) we conclude that

$$\int_\omega v_x^2 \le \liminf \mathcal{F}_\epsilon(v_\epsilon) \le \liminf \mathcal{F}_\epsilon(\phi_\epsilon) = \int_\omega \phi_x^2,$$

verifying (32).

STEP 4. Finally, we claim that the passage to a subsequence at the beginning of Step 1 was unnecessary, and $v_\epsilon \to v$ strongly in H^1 on any compact subset of ω. Indeed, we observed at the beginning of the proof that the solution of (26) is unique. So the family $\{v_\epsilon\}$ has just one limit point, namely the unique minimizer of the limiting variational problem. Moreover the convergence is strong in H^1 on compact sets, as an easy consequence of the fact that the energies converge. □

The preceding theorem assumes that $\epsilon^2/\gamma_\epsilon \to 0$ and obtains a limiting variational problem with no potential term, whose minimizer is unique. If instead $\epsilon^2/\gamma_\epsilon$ converges to a nonzero value β, we can still argue similarly but the limiting variational problem has a potential term and its minimizer is not necessarily unique.

Theorem 3. *Suppose ϵ and $\delta = \delta(\epsilon) \to 0$ with*

$$\frac{\delta}{\epsilon} \to 0 \quad \text{and} \quad \frac{\epsilon^2}{\gamma_\epsilon} \to \beta > 0.$$

Then (after possibly passing to a subsequence) the rescaled profiles v_ϵ of our local minimizers u_ϵ converge to a minimizer of

$$\min_{v\in\mathcal{A}} \int_\omega |v_x|^2 + \beta(v^2-1)^2 \tag{35}$$

where the admissible set is again defined by (27). The convergence is strong in H^1 on compact subsets of ω.

The proof is a minor modification of the one given for Theorem 2, so it can safely be left to the reader.

Remark: The asymptotic variational problem (35) can have more than one minimizer. For example, this occurs for sufficiently large β if the neck profile f is symmetric about 0 with a local maximum at $x = 0$. To see why, note that as $\beta \to \infty$, the asymptotic variational problem is minimized by putting a transition layer at a (global) minimum of the neck profile. If $f(x) = f(-x)$ with a maximum at 0, then the transition layer of the optimal v is not at 0 and $-v(-x)$ (which has the same energy) is another minimizer.

Remark: Theorem 3 does not assert that *every* minimizer of the asymptotic problem is a limit of geometrically constrained walls. We wonder whether such a statement might be true.

In conclusion: when the neck is thin ($\delta \ll \epsilon$) our geometrically constrained walls are essentially one-dimensional, in agreement with Bruno's ansatz (2). Besides determining the limiting wall profile, we have also determined the scaling of the wall energy. In fact, the preceding results imply that

$$\lim \left(\frac{\epsilon}{\gamma_\epsilon \delta^2}\right) F_\epsilon(u_\epsilon) \to \text{min value of the asymptotic variational problem}$$

where the right hand side is given by either (26) or (35), depending on the behavior of $\epsilon^2/\gamma_\epsilon$.

4.2. A normal neck

We assume throughout this subsection that

$$\delta/\epsilon \to \alpha$$

with $0 < \alpha < \infty$. (The thin and thick necks are essentially the cases $\alpha = 0$ and $\alpha = \infty$ respectively.) It is convenient to use a scaling slightly different from that of Section 4.1, namely:

$$x \to x/\epsilon, \quad y \to y/\epsilon, \quad z \to z/\epsilon.$$

The rescaled functional is therefore

$$\mathcal{F}_\epsilon(v) = \frac{1}{\gamma_\epsilon \epsilon} F_\epsilon(u) = \int_{\omega_\epsilon} |\nabla v|^2 + \frac{\epsilon^2}{\gamma_\epsilon} \int_{\omega_\epsilon} (v^2 - 1)^2 \tag{36}$$

where

$$v(x, y, z) = u(x\epsilon, y\epsilon, z\epsilon) \tag{37}$$

and ω_ϵ is the image of Ω_ϵ under this change of variables. As before, we let ω be the limiting rescaled domain. It has the form

$$\omega = \omega^l \cup \rho \cup \omega^r$$

where $\omega^l = \{x < -1\}$ and $\omega^r = \{x > 1\}$ are half-spaces and the rescaled neck is $\rho = \{\sqrt{y^2+z^2} < \alpha f(x),\ |x| \le 1\}$ (see Figure 6).

Our goal is to characterize the limiting behavior of v_ϵ, defined now as the image of our local minimizer u_ϵ under the scaling (37). To begin, we observe that the rescaled energy is uniformly bounded provided $\epsilon^2/\gamma_\epsilon \le C$. This follows from Lemma 3, since the rescaled functional (36) under consideration here is exactly $(\delta/\epsilon)^2$ times the one considered in the last section.

The asymptotic behavior is easy to understand heuristically. If $\delta/\epsilon \to \alpha$ and $\epsilon^2/\gamma_\epsilon \to 0$ the domain of integration in (36) converges to ω and the potential term becomes irrelevant. Therefore the limiting wall profile solves $\Delta v = 0$ in ω, with the natural boundary condition $\partial v/\partial n = 0$ at $\partial\omega$, and the additional condition "at infinity" that v tend to ± 1 as $x \to \pm\infty$.

To prove this result, we need a scheme for imposing the "conditions at infinity" variationally. We'll do this by requiring that $u \in \mathcal{A}$ where

$$\mathcal{A} = \{v \in H^1_{loc}(\omega) \,:\, v - \chi_{\omega^r} + \chi_{\omega^l} \in L^6(\omega)\}. \tag{38}$$

Here χ_A is the characteristic function of the set A, so

$$\chi_{\omega^r} - \chi_{\omega^l} = \begin{cases} -1 \text{ in } \omega^l \\ \ \ 0 \ \text{ in } \rho \\ \ \ 1 \ \text{ in } \omega^r. \end{cases}$$

This scheme is convenient, because the scale-invariant Poincare-type inequality (30) makes it easy to prove that $\lim v_\epsilon$ is in $\mathcal{A}$.

Here is the analogue of Theorem 2 for a normal neck.

Theorem 4. *Suppose ϵ and $\delta = \delta(\epsilon) \to 0$ with*

$$\frac{\delta}{\epsilon} \to \alpha \quad \textit{and} \quad \frac{\epsilon^2}{\gamma_\epsilon} \to 0$$

for some $0 < \alpha < \infty$. Then the rescaled profiles v_ϵ of our local minimizers u_ϵ converge to the minimizer of

$$\min_{v\in\mathcal{A}} \int_\omega |\nabla v|^2 \tag{39}$$

where the admissible set is defined by (38). The convergence is strong in H^1 on compact subsets of ω.

Proof: The proof is entirely parallel to that of Theorem 2, so we shall be relatively brief. Notice that as before, the minimizer of the limit problem (39) is unique, as a consequence of convexity.

STEP 1. *After passing to a subsequence if necessary, v_ϵ converges weakly to some limit $v \in H^1_{\text{loc}}(\omega)$*, with

$$\int_\omega |\nabla v|^2 \le \liminf_\epsilon \int_{\omega_\epsilon} |\nabla v_\epsilon|^2.$$

The argument is entirely parallel to what we did earlier.

STEP 2. *The limit is admissible,* i.e. $v \in \mathcal{A}$. The argument is again parallel to the proof of Theorem 2. Arguing as for (31) but taking into account the different scaling we get a uniform bound on $\|v_\epsilon - \overline{v_\epsilon}\|_{L^6(\omega^l_\epsilon)}$, where $\overline{v_\epsilon}$ is the mean of v_ϵ on ω^l_ϵ. Part (b) of Theorem 1 tells us that $\overline{v_\epsilon} \to -1$. Arguing as in Theorem 2, we conclude that for any compact $\omega_b \subset \omega^l$ we have

$$\int_{\omega_b} |v_\epsilon + 1|^6 \leq C.$$

It follows (taking $\epsilon \to 0$ then $\omega_b \uparrow \omega^l$) that

$$\|v + 1\|_{L^6(\omega^l)} \leq C.$$

The same argument shows that $\|v - 1\|_{L^6(\omega^r)} \leq C$, and together these give $v \in \mathcal{A}$.

STEP 3. *The limit v solves the asymptotic variational problem* (39). Indeed, consider any $\phi \in \mathcal{A}$. As before, we may assume without loss of generality that

$$|\phi| \leq 1$$

since truncating ϕ above by 1 and below by -1 maintains admissibility and decreases the value of $\int |\nabla \phi|^2$. We define a test function for F_ϵ by rescaling ϕ:

$$\psi_\epsilon(x, y, z) = \phi(x/\epsilon, y/\epsilon, z/\epsilon).$$

(To be sure ψ_ϵ is defined everywhere in Ω_ϵ, ϕ should be defined in a domain slightly larger than ω, obtained by expanding slightly the neck region ρ. Such an extension exists, with control on the L^∞ and H^1 norms, since ω is a Lipschitz domain.)

The function ψ_ϵ is admissible for (20) for sufficiently small ϵ. In fact

$$\|\psi_\epsilon - u_{0,\epsilon}\|_{L^2(\Omega_\epsilon)} \leq C\|\psi_\epsilon - u_{0,\epsilon}\|_{L^6(\Omega_\epsilon)}$$

with C independent of ϵ, since the volume of Ω_ϵ is uniformly bounded and $|\psi_\epsilon - u_{0,\epsilon}| \leq 2$. The right hand side tends to 0 with ϵ, since

$$\int_{\Omega_\epsilon} |\psi_\epsilon - u_{0,\epsilon}|^6 \leq \epsilon^3 \left(\int_{\omega^l} |\phi + 1|^6 + \int_{\omega^r} |\phi - 1|^6 + \int_{\text{nbhd of } \rho} |\phi|^6 \right) \leq C\epsilon^3,$$

using the fact that $\phi \in \mathcal{A}$. It follows from the definition (20) of u_ϵ that $F(u_\epsilon) \leq F(\psi_\epsilon)$. In the rescaled variables this says

$$\mathcal{F}_\epsilon(v_\epsilon) \leq \mathcal{F}_\epsilon(\phi),$$

which is the analogue of (34). Proceding as for Theorem 2, we conclude that

$$\int_\omega |\nabla v|^2 \leq \int_\omega |\nabla \phi|^2.$$

Thus v is a minimizer of the limit problem.

STEP 4. *The passage to a subsequence was unnecessary, and the convergence is strong in H^1 on compact subsets of ω.* The justification is the same as before. □

As in the case of the thin neck, our argument also works when $\epsilon^2/\gamma_\epsilon \to \beta$ but the limiting variational problem has a potential term and its minimizer is not necessarily unique. As a result we cannot conclude that the entire family v_ϵ converges.

Theorem 5. *Suppose ϵ and $\delta = \delta(\epsilon) \to 0$ with*

$$\frac{\delta}{\epsilon} \to \alpha \quad \textit{and} \quad \frac{\epsilon^2}{\gamma_\epsilon} \to \beta$$

where $0 < \alpha, \beta < \infty$. Then (after possibly passing to a subsequence) the rescaled profiles v_ϵ of our local minimizers u_ϵ converge to a minimizer of

$$\min_{v \in \mathcal{A}} \int_\omega |\nabla v|^2 + \beta(v^2 - 1)^2 \tag{40}$$

where the admissible set is again defined by (38). The convergence is strong in H^1 on compact subsets of ω.

The proof is a minor modification of the one given for Theorem 4, so it can safely be left to the reader.

In conclusion: for a normal neck ($\delta \sim \epsilon$) our geometrically constrained walls are *not* one-dimensional, and they are *not* confined to the neck region. Their profile is determined by solving a 3D variational problem, which depends in an essential way on the shape of the neck. Besides determining the character of the wall we have also determined its energy:

$$\lim \left(\frac{1}{\gamma_\epsilon \epsilon}\right) F_\epsilon(u_\epsilon) \to \text{min value of the asymptotic variational problem}$$

where the right hand side is given by either (39) or (40), depending on the behavior of $\epsilon^2/\gamma_\epsilon$.

4.3. A thick neck

For a thick neck the analysis is more or less the same, but the outcome is different. The limiting domain depends only on the inner radius of the neck, i.e. on

$$f_{\min} = \min_{|x| \le 1} f.$$

As a result, the behavior of the wall is virtually independent of the neck profile. The physical reason is simple: for a thick neck the wall resides almost entirely *outside* the neck. This would be obvious if we took $\epsilon = 0$ and $\delta > 0$; our analysis will show it also holds whenever

$$\delta/\epsilon \to \infty$$

(which we assume throughout this subsection).

The appropriate scaling in this setting is

$$x \to x/\delta, \quad y \to y/\delta, \quad z \to z/\delta.$$

The associated rescaled functional is

$$\mathcal{F}_\epsilon(v) = \frac{1}{\gamma_\epsilon \delta} F_\epsilon(u) = \int_{\omega_\epsilon} |\nabla v|^2 + \frac{\delta^2}{\gamma_\epsilon} \int_{\omega_\epsilon} (v^2 - 1)^2 \tag{41}$$

where

$$v(x, y, z) = u(x\delta, y\delta, z\delta) \tag{42}$$

and ω_ϵ is the image of Ω_ϵ under this change of variables. As usual, we let ω be the limiting rescaled domain. Its form is different from before (see Figure 7). Briefly, $\omega \subset \mathbf{R}^3$ is the domain obtained by glueing the right half-plane $x > 0$ to the left half-plane $x < 0$ along a disk of radius $f_{\min}$. More formally:

$$\omega = \omega^l \cup \Gamma \cup \omega^r$$

where

$$\omega^l = \{x < 0\}, \quad \omega^r = \{x > 0\},$$

and

$$\Gamma = \{x = 0, \sqrt{y^2 + z^2} \leq f_{\min}\}.$$

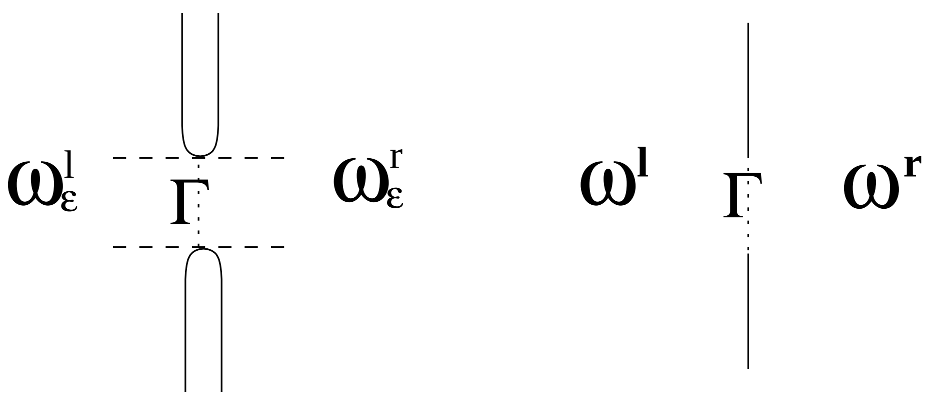

Fig. 7. *The limiting rescaled domain for a thick neck.*

Our goal, as usual, is to characterize the limiting behavior of v_ϵ, defined now as the image of our local minimizer u_ϵ under the scaling (42). Since $\epsilon \ll \delta$, our standing hypothesis $\max\{\epsilon^2, \delta^2\}/\gamma_\epsilon \leq C$ says $\delta^2/\gamma_\epsilon \leq C$, and Lemma 2 shows that the rescaled energy is uniformly bounded. The asymptotic behavior is easy to understand heuristically. In the extreme $\delta^2/\gamma_\epsilon \to 0$, when the potential term is irrelevant, the limiting wall profile solves $\Delta v = 0$ in ω with the natural boundary condition $\partial v/\partial n = 0$ at both sides of the "crack" $\{x = 0, \sqrt{y^2 + z^2} > f_{\min}\}$, and with the conditions "at infinity" that v tend to ± 1 as $x \to \pm\infty$,

As in the case of a normal neck, we impose the far-field condition that $v \to \pm 1$ variationally, by restricting the limit problem to the the "admissible set"

$$\mathcal{A} = \{v \in H^1_{loc}(\omega) \, : \, v - \chi_{\omega^r} + \chi_{\omega^l} \in L^6(\omega)\}. \tag{43}$$

Here is the analogue of Theorem 2 for a thick neck.

Theorem 6. *Suppose* ϵ *and* $\delta = \delta(\epsilon) \to 0$ *with*

$$\frac{\delta}{\epsilon} \to \infty \quad \textit{and} \quad \frac{\delta^2}{\gamma_\epsilon} \to 0.$$

Then the rescaled profiles v_ϵ *of our local minimizers* u_ϵ *converge to the minimizer of*

$$\min_{v \in \mathcal{A}} \int_\omega |\nabla v|^2 \tag{44}$$

where the admissible set is defined by (43). The convergence is strong in H^1 *on compact subsets of* ω.

Proof: Notice that the minimizer of (44) is unique, by convexity. This implies that it's an odd function of x (otherwise $\frac{1}{2}v(x,y,z) - \frac{1}{2}v(-x,y,z)$ would be another minimizer). So v can alternatively be found by minimizing $\int |\nabla v|^2$ on the halfspace $x > 0$, with boundary condition $v = 0$ on Γ.

Our overall strategy is the same as in the other cases. At finite ϵ the rescaled domain ω_ϵ consists of the rescaled neck $\delta^{-1}R_\epsilon$ connecting the rescaled left side ω^l_ϵ and the rescaled right side ω^r_ϵ. The domains ω^l_ϵ and ω^r_ϵ are contained in the half-spaces $\{x < -\epsilon/\delta\}$ and $\{x > \epsilon/\delta\}$ respectively. As we pass to the limit, we prefer to work as much as possible in the fixed (limiting) halfspaces ω^l and ω^r. Therefore it is convenient to define

$$v^l_\epsilon(x,y,z) = v_\epsilon(x - \epsilon/\delta, y, z), \quad v^r_\epsilon(x,y,z) = v_\epsilon(x + \epsilon/\delta, y, z)$$

which are defined on $\delta^{-1}\Omega^l$ and $\delta^{-1}\Omega^r$ respectively. The rescaled energy can then be expressed as

$$\mathcal{F}(v_\epsilon) = \int_{\delta^{-1}\Omega^l} |\nabla v^l_\epsilon|^2 + \int_{\delta^{-1}\Omega^r} |\nabla v^r_\epsilon|^2 + \int_{\delta^{-1}R_\epsilon} |\nabla v_\epsilon|^2 + \frac{\delta^2}{\gamma_\epsilon} \int_{\omega_\epsilon} (v_\epsilon^2 - 1)^2.$$

STEP 1. *After passing to a sequence if necessary, v^l_ϵ and v^r_ϵ converge weakly to limits v^l and v^r, with*

$$\int_{\omega^l} |\nabla v^l|^2 + \int_{\omega^r} |\nabla v^r|^2 \le \liminf_\epsilon \int_{\omega_\epsilon} |\nabla v_\epsilon|^2.$$

The argument is similar to the corresponding steps of Theorems 2 and 4.

STEP 2. *The limit is admissible, i.e. the function*

$$v = \begin{cases} v^l \text{ in } \omega^l \\ v^r \text{ in } \omega^r \end{cases}$$

belongs to the admissible set $\mathcal{A}$. To prove that $v - \chi_{\omega^r} + \chi_{\omega^l} \in L^6(\omega)$ we have only to repeat the arguments used for Theorems 2 and 4. However we must also show that v is continuous across Γ. The argument uses the uniform bound

$$\int_{\delta^{-1}R_\epsilon} |\nabla v_\epsilon|^2 \le C,$$

and the fact that $\delta^{-1}R_\epsilon$ contains the cylinder $(-\epsilon/\delta, \epsilon/\delta) \times \Gamma$. By the fundamental theorem of calculus, for any $(y,z) \in \Gamma$ we have

$$|v_\epsilon(-\epsilon/\delta, y, z) - v_\epsilon(\epsilon/\delta, y, z)| \le \int_{-\epsilon/\delta}^{\epsilon/\delta} |v_x| dx \le (2\epsilon/\delta)^{1/2} \left(\int_{-\epsilon/\delta}^{\epsilon/\delta} |v_x|^2 dx \right)^{\frac{1}{2}}.$$

Therefore

$$\int_\Gamma |v_\epsilon(-\epsilon/\delta, y, z) - v_\epsilon(\epsilon/\delta, y, z)|^2 \, dy \, dz \le C \frac{\epsilon}{\delta} \to 0.$$

In other words

$$v^l_\epsilon|_{x=0} - v^r_\epsilon|_{x=0} \tag{45}$$

converges to 0 in $L^2(\Gamma)$ as $\epsilon \to 0$. Passing to the limit, it follows that v is continuous across Γ, as desired. (We have used the fact that the L^2 norm of (45) on Γ is lower semicontinuous under weak H^1_{loc} convergence, since $v \mapsto v^l_\epsilon\big|_{x=0} - v^r_\epsilon|_{x=0}$ is continuous as a map from $H^1_{loc} \to L^2$.)

STEP 3. *The limit v solves the asymptotic variational problem* (44). The proof follows the usual pattern. It suffices, by truncation, to consider $\phi \in \mathcal{A}$ satisfying $|\phi| \le 1$. The associated test function for F_ϵ is

$$\psi_\epsilon(x,y,z) = \phi(x/\delta, y/\delta, z/\delta).$$

Note that ψ_ϵ is defined everywhere on Ω_ϵ. It is admissible for (20 because $\int_{\Omega_\epsilon} |\psi_\epsilon - u_{0,\epsilon}|^2$ tends to 0, by the same argument used for Step 3 of Theorem 4. Therefore $F(u_\epsilon) \le F(\psi_\epsilon)$, or equivalently

$$\mathcal{F}_\epsilon(v_\epsilon) \le \mathcal{F}_\epsilon(\phi).$$

Proceding as for Theorems 2 and 4, it follows that

$$\int_\omega |\nabla v|^2 \leq \int_\omega |\nabla \phi|^2.$$

Thus v is a minimizer of the limit problem.

STEP 4. *The passage to a subsequence was unnecessary, and the convergence is strong in H^1 on compact subsets of ω.* The justification is the same as before. □

As in the prior cases, our argument also works when $\epsilon^2/\gamma_\epsilon \to \beta$ but the limiting variational problem has a potential term and is therefore nonconvex. We conjecture that its solution is unique, but we have not proved this, so the following theorem asserts only convergence of a subsequence.

Theorem 7. *Suppose ϵ and $\delta = \delta(\epsilon) \to 0$ with*

$$\frac{\delta}{\epsilon} \to \infty \quad \textit{and} \quad \frac{\epsilon^2}{\gamma_\epsilon} \to \beta$$

where $0 < \beta < \infty$. Then (after possibly passing to a subsequence) the rescaled profiles v_ϵ of our local minimizers u_ϵ converge to a minimizer of

$$\min_{v \in \mathcal{A}} \int_\omega |\nabla v|^2 + \beta(v^2 - 1)^2 \tag{46}$$

where the admissible set is again defined by (43). The convergence is strong in H^1 on compact subsets of ω.

The proof is left to the reader.

In conclusion: for a thick neck ($\delta \gg \epsilon$) our geometrically constrained walls spread far beyond the neck – indeed, the associated "exchange energy" is located almost entirely in the bulk. The profile depends on the minimum aperture of the neck, but not on the rest of its shape. As in the other cases, besides determining the wall profile we have also determined the wall energy:

$$\lim \left(\frac{1}{\gamma_\epsilon \delta}\right) F_\epsilon(u_\epsilon) \to \text{min value of the asymptotic variational problem}$$

where the right hand side is given by either (44) or (46), depending on the behavior of $\epsilon^2/\gamma_\epsilon$.

5. Remarks

We have restricted our attention to axially-symmetric necks in 3D domains. Our method extends straightforwardly however to non-axially-symmetric necks and higher-dimensional problems. It also extends easily to some problems involving vector-valued unknowns. The analogous 2D problem, however, is different.

5.1. Generalizations

We formulated the problem in Section 2 assuming axial symmetry. This was however just for the sake of simplicity and clarity. Our analysis made no essential use of this symmetry.

We have focused on domains in $\mathbf{R}^3$ both for clarity and because the motivation involves magnetic point contacts. However our method can also be applied in higher dimensions, e.g. for solids of revolution in $\mathbf{R}^n$ for $n > 3$. In fact, the dimension enters our analysis in two places:

(i) The proof that the test function ψ_ϵ is in B_ϵ, i.e. that $\psi_\epsilon + 1$ tends to zero in L^2 on the left half of the unscaled domain, and similarly for $\psi_\epsilon - 1$ on the right half.
(ii) The proof that the limit $v = \lim v_\epsilon$ is admissible, i.e. that $v + 1 \in L^6$ on the left half of the rescaled domain, and similarly for $v - 1$ on the right half.

The argument for (i) works in any dimension $n \geq 2$, since its heart is the change of variables formula: if $g_\epsilon(x) = g(x/\epsilon)$ and $D \subset \mathbf{R}^n$,

$$\int_D g_\epsilon^2 = \epsilon^n \int_{\epsilon^{-1} D} g^2.$$

The argument for (ii) also works in dimension $n \geq 3$, since its heart is the scale-invariant Poincaré-type inequality

$$\left(\int_D |u - \overline{u}|^q \right)^{1/q} \leq C \left(\int_D |\nabla u|^2 \right)^{1/2} \quad \text{with } q = \tfrac{2n}{n-2}. \tag{47}$$

Here is a natural vector-valued extension of our problem, motivated by micromagnetics. Consider the variational problem

$$\gamma_\epsilon \int_{\Omega_\epsilon} |\nabla m|^2 + \int_{\Omega_\epsilon} m_1^2 + m_2^2, \tag{48}$$

where $m = (m_1, m_2, m_3)$ is constrained to take values in the unit sphere $|m| = 1$ and $\Omega_\epsilon \subset \mathbf{R}^3$ is an axially-symmetric domain with a sharp neck as formulated in Section 2. Though the unknown is vector-valued the situation is very similar to our scalar problem (8), since m has two preferred values $(0, 0, \pm 1)$. In the scalar case we used truncation to restrict attention to $|u_\epsilon| \leq 1$ and $|\phi| \leq 1$; in the vector-valued setting such truncation isn't possible – but it isn't necessary either, since we have the uniform bound $|m| = 1$. The rest of our analysis used only Sobolev estimates, and exactly the same arguments work in the vector-valued setting (48).

5.2. Two space dimensions

Our comments on the extension to $\mathbf{R}^n$ make it clear that the planar case is different. The main problem is the lack of a scale-invariant estimate like (47) in $\mathbf{R}^2$. This is not just a technical problem: it reflects the slow decay of solutions to Laplace's equation in two space dimensions. For example, in the case of a thick neck the limiting profile should solve $\Delta v = 0$ in the half-plane $x > 0$, with $v = 0$ on an interval Γ along the boundary, $\partial v / \partial n = 0$ on the rest of the boundary, and $v \to 1$ as $x \to \infty$.

Upon examination, there is a setting where arguments like those in the present paper can be made to work. This is the case when $\delta/\epsilon^a \to 0$ for some $a > 1$ – an *extremely* thin neck so to speak.

We conjecture, however, that all our main conclusions remain valid in 2D: geometrically constrained walls should exist; their profiles should be one-dimensional when $\delta \ll \epsilon$; and their profiles should be independent of the neck shape when $\delta \gg \epsilon$.

References

1. A. Braides, Γ-convergence for beginners, Oxford Lecture Series in Mathematics and its Applications **22**, 2002.
2. P. Bruno, Geometrically constrained magnetic wall, Phys. Rev. Lett. **83** (1999), 2425–2428.
3. E. Cabib, L. Freddi, A. Morassi, D. Percivale, Thin notched beams, J. Elasticity **64** (2002), 157–178.
4. J. Casado-Díaz, M. Luna-Layez, F. Murat, Asymptotic behavior of diffusion problems in a domain made of two cylinders of different diameters and heights, C. R. Acad. Sci. Paris, Série I **338** (2004), 133–138.
5. J. Casado-Díaz, M. Luna-Layez, F. Murat, Asymptotic behavior of an elastic beam fixed on a small part of one of its extremities, C. R. Acad. Sci. Paris, Série I **338** (2004), 975–980.
6. J. Casado-Díaz, M. Luna-Layez, F. Murat, The diffusion equation in a notched beam, to appear.
7. T. DelVecchio, The thick Neumann's sieve, Ann. Mat. Pura Appl. **147** (1987), 363–402.
8. N. García, M. Muñoz, Y.-W. Zhao, Magnetoristance in excess of 200% in ballistic Ni nanocontacts at room temperature and 100 Oe, Phys. Rev. Lett. **82** (1999), 2923–2926.
9. J.K.Hale, J.Vegas, A nonlinear parabolic equation with varying domain, Arch. Rat. Mech. Anal. **86** (1984), 99–123.
10. S. Jimbo, Singular perturbation of domains and semilinear elliptic equation, J. Fac. Sci. Univ. Tokyo **35** (1988), 27–76.
11. S. Jimbo, Singular perturbation of domains and semilinear elliptic equation 2, J. Diff. Equat. **75** (1988), 264–289.
12. R.V. Kohn, P. Sternberg, Local minimizers and singular perturbations, Proc. Roy. Soc. Edinburgh **111 A** (1989), 69–84.
13. L. Modica, S. Mortola, Un esempio di Γ-convergenza, (Italian) Boll. Un. Mat. Ital. B (5) **14** (1977), 285–299.
14. V.A. Molyneux, V.V. Osipov and E.V. Ponizovskaya, Stable two- and three-dimensional geometrically constrained magnetic structures: The action of magnetic fields, Phys. Review B **65** (2002), 184425.
15. J. Rubinstein, M. Schatzman and P. Sternberg, Ginzburg-Landau model in thin loops with narrow constrictions, SIAM J. Appl. Math. **64** (2004), 2186–2204.