Effective dynamics for ferromagnetic thin films: a rigorous justification

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Abstract

In a thin-film ferromagnet, the leading-order behavior of the magnetostatic energy is a strong shape anistropy, penalizing the out-of-plane component of the magnetization distribution. We study the thin-film limit of Landau-Lifshitz-Gilbert dynamics, when the magnetostatic term is replaced by this local approximation. The limiting 2D effective equation is overdamped, i.e. it has no precession term. Moreover if the damping coefficient of 3D micromagnetics is α then the damping coefficient of the 2D effective equation is $\alpha + 1/\alpha$; thus reducing the damping in 3D can actually increase the damping of the effective equation. This result was previously shown by Garcia-Cervera and E using asymptotic analysis; our contribution is a mathematically rigorous justification.

1 Introduction

Landau-Lifshitz-Gilbert dynamics governs the evolution of magnetization in a ferromagnetic body. It is a damped Hamiltonian system, describing the combined effect of gyromagnetic precession and damping.

Our attention is on ferromagnetically soft thin film elements. Such films – made for example of permalloy – are relatively easy to manufacture. They are used in many devices, and have been explored at length experimentally and numerically.

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There are three distinct parameters with the dimensions of length, namely

$$\begin{aligned} t &= \text{the film thickness,} \\ l &= \text{the in-plane diameter, and} \\ w &= \text{the exchange length of the ferromagnetic material.} \end{aligned}$$

Thus there are two nondimensional parameters,

$$\epsilon = \frac{t}{l} = \text{aspect ratio}, \qquad d = \frac{w}{l} = \text{normalized exchange length},$$

and a variety of different thin-film regimes [2, 8, 4, 5, 11, 13]. Since the aspect ratio is a small parameter, it is natural to seek a 2D effective equation, expressing the asymptotic dynamics in the limit as the aspect ratio tends to 0.

The magnetostatic energy is nonlocal, but in certain parameter regimes its leading-order behavior is local. We focus on one such regime, namely

$$\epsilon|\log \epsilon| \ll d^2 \ll 1, \tag{1}$$

and we consider only the simplified problem where the magnetostatic energy is replaced by a penalization term proportional to $\int_\omega m_3^2\, dxdy$. We assume moreover that the nondimensional Gilbert damping parameter α is of order 1.

This regime – more precisely, a closely related one – was recently addressed by Garcia-Cervera and E using asymptotic analysis [7]. They showed that the 3D Landau-Lifshitz-Gilbert (LLG) equation

$$m_t = m \times H - \alpha m \times (m \times H)$$

yields a 2D effective equation

$$m'_\tau = -(\alpha + \frac{1}{\alpha}) m' \times (m' \times H').$$

Here $H = -\nabla_m E$ is the effective field; τ is the time-scale of the effective dynamics; and $m' = (m_1, m_2)$ since the effective equation has $m_3 = 0$. We shall review their argument in Section 2. The conclusion is remarkable: it says that the 3D precessional term $m \times H$ gets converted, in the thin-film limit, to a 2D damping term proportional to $1/\alpha$. In particular, if $\alpha < 1$ then a decrease of the 3D damping actually increases the 2D effective damping. Briefly, the intuition is as follows: in the thin-film limit the energy strongly prefers m to be in-plane. Precession pushes it out-of-plane, while damping brings it back. Since $m_3 \approx 0$ these two effects must roughly balance, so precession becomes a damping term.

The goal of the present paper is a rigorous study of this phenomenon. The task is subtle, because solutions of the LLG equation are not known to be regular. We must deal with weak solutions, satisfying relatively weak energy-based estimates. This makes it difficult to find the limit of a nonlinear term like $m \times H$. The resolution of this difficulty uses the specific form of the equation.

We wish to study the observation of Garcia-Cervera and E in the simplest possible setting. That is why we study only a simplified problem, where the magnetostatic energy is replaced with a penalization term proportional to $\int_\omega m_3^2$. In doing so, we capture the essential physics of the regime (1), where exchange energy is dominant and the thin-film-limit does not involve vortices [11]. It would be interesting to prove similar results for $\epsilon \sim d^2 \ll 1$, the scaling considered in [7]. This problem seems more subtle, however, since the associated thin-film-limit involves vortices [13].

Our regime is different from the one considered in [2, 4]. Those authors consider the thin-film limit of LLG when $\epsilon \to 0$ with d held fixed, of order 1. In this case the exchange and (leading-order) magnetostatic terms interact, i.e. the asymptotic energy looks like $\epsilon \int_\omega |\nabla m|^2 + \epsilon \int_\omega m_3^2$. So the evolution of m is not mainly in-plane, and the effective equation is entirely different.

2 Motivation and heuristics

This section gives a brief introduction to thin-film micromagnetics, reviews the asymptotic analysis of Garcia-Cervera and E, and explains why our local approximation to the micromagnetic energy captures the essential physics of the regime (1).

2.1 Landau-Lifshitz-Gilbert dynamics

The evolution of the magnetization vector m is described by the Landau-Lifshitz-Gilbert (LLG) equation

$$\frac{\partial m}{\partial t} = m \times H(m) - \alpha m \times (m \times H(m)) \quad \text{in } \Omega \tag{2}$$

with boundary condition

$$\frac{\partial m}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

The right side of (2) has two terms: gyromagnetic $m \times H(m)$ and damping $m \times (m \times H(m))$. The gyromagnetic term describes the precession induced by the "effective field" $H(m)$; it can be obtained from quantum-mechanical principles. The damping term is more phenomenological – in other words, it is not derived from an atomic-scale model – but it is widely accepted, based on the match between experimental observations and numerical simulation. The sign convention of the precession term is arbitrary – some authors use $-m \times H(m)$ rather than $m \times H(m)$ – since the physics does not distinguish between a right-handed and left-handed cross-product; the sign of the damping term, however, is essential. The dimensionless parameter α in front of the damping term is material-dependent, and usually rather small – on the order of .01-0.1.

The effective field $H(m) = -\delta E/\delta m$ is, up to sign, the first-variation of the micromagnetic energy

$$E(m) = \frac{d^2}{2} \int_\Omega |\nabla m|^2 + Q \int_\Omega \phi(m) + \frac{1}{2} \int_{\mathbf{R}^3} |\nabla u|^2 - \int_\Omega h_{ext} \cdot m. \tag{3}$$

The four terms are known as the exchange, anisotropy, magnetostatic, and external (or Zeeman) energies respectively. The domain $\Omega \subset \mathbf{R}^3$ is the region occupied by the ferromagnet; $m \colon \Omega \to \mathbf{R}^3$ is the direction of magnetization, constrained by

$$|m(x)| = 1 \quad \text{for } x \in \Omega \tag{4}$$

and understood to equal 0 outside Ω; and h_{ext} is the external, applied field. The function u, defined on all $\mathbf{R}^3$, is defined by

$$\operatorname{div}(\nabla u + m) = 0 \quad \text{in } \mathbf{R}^3, \tag{5}$$

in the sense of distributions. Thus the magnetostatic energy is nonlocal in m. Its physical interpretation involves the long-range interaction of magnetic dipoles; mathematically it can be viewed as a penalization favoring $\operatorname{div} m = 0$, since ∇u is the Helmholtz projection of m onto gradients. It can be expressed as an integral over Ω alone, since

$$\int_{\mathbf{R}^3} |\nabla u|^2 = -\int_\Omega m \cdot \nabla u = \int_\Omega h_{ind} \cdot m,$$

where

$$h_{ind} = -\nabla u$$

is the magnetic field induced by m. The nonlocal character of this term makes it onerous to evaluate, and a major stumbling block to numerical simulation. For more information on micromagnetics see [1], [9].

2.2 The analysis of Garcia-Cervera and E

In this section we review the asymptotic analysis done by Garcia-Cervera and E in [7]. They focus on the regime $d^2 \sim \epsilon$, where the exchange energy is comparable to the magnetostatic energy associated with nonzero $\operatorname{div} m$. The distinction between their regime and ours will be explained in Section 2.3.

When discussing thin films, it is convenient to scale all spatial variables by the in-plane diameter l of the film. After this nondimensionalization, the domain of the ferromagnet is $\Omega_\epsilon = \omega \times (0, \epsilon)$ where ω is the rescaled cross-section and $\epsilon = t/l$ is the aspect ratio.

We need the leading-order behavior of the magnetostatic energy. To explain it we assume – *for this section only* – that $m = (m_1, m_2, m_3)$ is independent of the thickness variable. (Our rigorous analysis, presented in Section 3, makes no such hypothesis.) Then the magnetostatic energy can be written as

$$\int_{R^3} |\nabla u|^2 = \epsilon \int_{R^2} \frac{(\eta \cdot \hat{m}')^2}{|\eta|^2} (1 - \hat{\Gamma}_\epsilon(|\eta|))\, d\eta + \epsilon \int_{R^2} \hat{m}_3^2 \hat{\Gamma}_\epsilon(|\eta|)) d\eta \tag{6}$$

where $m' = (m_1, m_2)$, $\hat{f}$ is the Fourier transform of f, and

$$\hat{\Gamma}_\epsilon(|\eta|)) = \frac{1 - exp(-2\pi\epsilon|\eta|)}{2\pi\epsilon|\eta|}.$$

(This formula is well-known; a brief derivation can be found in Appendix B of [11].)

If the magnetization is smooth enough (specifically: if its Fourier transform lives predominantly at $|\eta| \ll \frac{1}{\epsilon}$) then the leading-order behavior of each term in (6) is

$$\int_{R^3} |\nabla u|^2 \approx \pi\epsilon^2 \int_{R^2} \frac{(\eta \cdot \hat{m}')^2}{|\eta|}\, d\eta + \epsilon \int_{R^2} \hat{m}_3^2\, d\eta. \tag{7}$$

With this approximation the full micromagnetic energy becomes

$$E(m) = \frac{d^2\epsilon}{2}\int_\omega |\nabla' m|^2 + Q\epsilon\int_\omega \phi(m) + \frac{\pi\epsilon^2}{2}\int_{R^2} \frac{(\eta\cdot\hat{m}')^2}{|\eta|}d\eta + \frac{\epsilon}{2}\int_\omega m_3^2 - \epsilon\int_\omega h_{ext}\cdot m. \quad (8)$$

If we assume $d^2 = \epsilon\tilde{d}^2$, $Q = \epsilon\tilde{Q}^2$, $h_{ext} = \epsilon\tilde{h}_{ext}$ then the exchange, anisotropy, nonlocal, and applied-field terms interact, while the term $\int_\omega m_3^2$ is more singular. So m_3 should be small compared to m', specifically $m_3 = \epsilon\tilde{m}_3$. Rescaling time by $\tilde{t} = \epsilon t$ and the effective field by $H = \epsilon\tilde{H}$, the LLG equation with the simplified magnetostatic energy (7) becomes

$$\epsilon\frac{\partial m_1}{\partial\tilde{t}} = -\epsilon^2\tilde{m}_3\tilde{H}_2 + \epsilon m_2\tilde{H}_3 + \alpha(\epsilon\tilde{H}_1 - \epsilon(m'\cdot\tilde{H}')m_1 - \epsilon^2\tilde{m}_3\tilde{H}_3 m_1)$$

$$\epsilon\frac{\partial m_2}{\partial\tilde{t}} = -\epsilon m_1\tilde{H}_3 + \epsilon^2\tilde{m}_3\tilde{H}_1 + \alpha(\epsilon\tilde{H}_2 - \epsilon(m'\cdot\tilde{H}')m_2 - \epsilon^2\tilde{m}_3\tilde{H}_3 m_2)$$

$$\epsilon^2\frac{\partial\tilde{m}_3}{\partial\tilde{t}} = -\epsilon m_2\tilde{H}_1 + \epsilon m_1\tilde{H}_2 + \alpha(\epsilon\tilde{H}_3 - \epsilon^2(m'\cdot\tilde{H}')\tilde{m}_3 - \epsilon^3\tilde{m}_3\tilde{H}_3\tilde{m}_3)$$

After collecting the leading order terms and assuming $\epsilon \ll \alpha$ we have

$$\tilde{H}_3 = \frac{1}{\alpha}(m_2\tilde{H}_1 - m_1\tilde{H}_2)$$

$$\frac{\partial m_1}{\partial\tilde{t}} = m_2\tilde{H}_3 + \alpha(\tilde{H}_1 - (m'\cdot\tilde{H}')m_1) \quad (9)$$

$$\frac{\partial m_2}{\partial\tilde{t}} = -m_1\tilde{H}_3 + \alpha(\tilde{H}_2 - (m'\cdot\tilde{H}')m_2) \quad (10)$$

Obviously $|m'| = 1$ to leading order, so we have

$$m_2\tilde{H}_3 = \frac{1}{\alpha}(\tilde{H}_1 - (m'\cdot\tilde{H}')m_1)$$

$$m_1\tilde{H}_3 = -\frac{1}{\alpha}(\tilde{H}_2 - (m'\cdot\tilde{H}')m_2).$$

Plugging this back to (9) and (10) we obtain

$$\frac{\partial m'}{\partial\tilde{t}} = -(\alpha + \frac{1}{\alpha})m' \times (m' \times \tilde{H}(m')) \quad \text{in } \omega$$

where $\tilde{H}(m') = -\frac{\delta\tilde{E}}{\delta m'}$ and

$$\tilde{E}(m') = \frac{\tilde{d}^2}{2}\int_\omega |\nabla m'|^2 + \tilde{Q}\int_\omega \phi(m') + \frac{\pi}{2}\int_{R^2}\frac{(\eta\cdot\hat{m}')^2}{|\eta|}d\eta - \int_\omega \tilde{h}_{ext}\cdot m'. \quad (11)$$

Thus the gyroscopic term becomes, in this thin-film-limit, a damping term proportional to $1/\alpha$.

What happened? The main point is that the energy associated with nonzero m_3 is more singular than the other terms. So $m_3 \approx 0$ and $m \approx (m', 0)$. Therefore (to summarize the argument less formally, but perhaps more transparently) the out-of-plane component of LLG gives

$$0 \approx (m \times H)_3 - \alpha(m \times (m \times H))_3 \approx (H', m'_\perp) + \alpha H_3$$

where $m'_\perp = (-m_2, m_1)$. In other words, $H_3 \approx -\frac{1}{\alpha}(H', m'_\perp)$. It follows that

$$\begin{aligned}(m \times H)' &\approx m' \times H_3 e_3 \\ &\approx -H_3 m'_\perp \\ &\approx \frac{1}{\alpha}(H', m'_\perp) m'_\perp \\ &\approx \frac{1}{\alpha}(H' - (H', m')m') \\ &\approx -\frac{1}{\alpha} m' \times (m' \times H').\end{aligned}$$

Thus the in-plane gyroscopic term is asymptotically a damping term.

2.3 Alternative regimes

In writing (7) as an approximation of the magnetostatic energy, we assumed that m' was sufficiently smooth. This assumption fails if $m' \cdot n' \neq 0$ at $\partial\omega$, since in this case m' has a jump discontinuity across the boundary and $\operatorname{div} m'$ has a singular contribution at $\partial\omega$. The associated energy scales like $\epsilon^2|\log\epsilon|$ times the boundary integral of $(m' \cdot n')^2$ [4, 11]. If we include this term, then (8) becomes

$$\begin{aligned}E(m) &= \frac{d^2\epsilon}{2}\int_\omega |\nabla' m|^2 + Q\epsilon \int_\omega \phi(m) + \frac{\epsilon^2}{2}\|(\operatorname{div} m)_{\text{smooth}}\|^2_{H^{-1/2}} \\ &\quad + \frac{\epsilon^2|\log\epsilon|}{4\pi}\int_{\partial\omega}(m' \cdot n')^2 + \frac{\epsilon}{2}\int_\omega m_3^2 - \epsilon\int_\omega h_{ext} \cdot m.\end{aligned}$$

The scaling of the anisotropy and Zeeman terms can be adjusted by letting Q and h_{ext} depend on ϵ. But the other four terms have no free parameters. When we choose to focus on a specific regime, we are effectively choosing which of these terms are important, and which are negligible.

Garcia-Cervera and E take $d^2 \sim \epsilon$. In this case the terms involving $\int_\omega m_3^2$ and $\int_{\partial\omega}(m' \cdot n')^2$ appear to become constraints, leaving the terms involving $|\nabla m|^2$ and $\operatorname{div} m$ as leading-order terms. But this cannot be quite right: if ω is simply-connected then there is no magnetization distribution satisfying $m_3 = 0$, $m' \cdot n' = 0$ and $\int_\omega |\nabla m|^2 < \infty$. Instead, m should develop vortices, and they should carry the leading-order contribution to the energy. Rigorous results are just beginning to emerge concerning energy minimization in this regime [13]. We expect – by analogy with other problems involving vortices (see e.g. [12, 10, 14]) – that rigorous analysis of the dynamics will be a fairly subtle matter.

But the conversion of the gyroscopic term to a damping term has nothing to do with with vortices. Rather, what matters is that the term involving $\int_\omega m_3^2$ act asymptotically as a constraint. Our regime $\epsilon|\log\epsilon| \ll d^2 \ll 1$ has this property, without any incentive for vortex formation. Indeed, since $\epsilon|\log\epsilon| \ll d^2$, the terms involving $\operatorname{div} m$ and $m' \cdot n'$ are negligible compared to exchange energy, so they can safely be ignored. But since $d^2 \ll 1$, the term involving m_3^2 is still singular, becoming asymptotically a constraint.

Let us take $d^2 = \delta \tilde{d}^2$, $Q = \delta \tilde{Q}$, and $h_{ext} = \delta \tilde{h}_{ext}$. Then our regime corresponds to $\epsilon|\log \epsilon| \ll \delta \ll 1$. Dropping the terms that should be negligible, we see that the energy divided by $\epsilon\delta$ is equivalent at leading order to

$$\frac{\tilde{d}^2}{2}\int_\omega |\nabla m'|^2 + \tilde{Q}\int_\omega \phi(m') + \frac{1}{2\delta}\int_\omega m_3^2 - \int_\omega \tilde{h}_{ext}\cdot m'$$

(with no condition that $m' \cdot n'$ vanish at $\partial\omega$). As we shall prove in the next section, the thin-film limit of LLG with this choice of energy has the expected behavior: the gyromagnetic term becomes, in the limit, a damping term with coefficient $1/\alpha$.

3 Rigorous analysis

We assume for simplicity that $\tilde{d} = 1$, $\tilde{Q} = 0$ and $\tilde{h}_{ext} = 0$. It is easier to deal with a fixed domain, so we rescale $\Omega_\epsilon = \omega \times (0, \epsilon)$ in the z direction, working instead on the domain $\Omega = \omega \times (0, 1)$. For the reasons explained above, we work with the (normalized) energy

$$E_\epsilon(m) = \frac{1}{2}\int_\Omega |\nabla' m|^2 + \frac{1}{2\epsilon^2}\int_\Omega (\frac{\partial m}{\partial z})^2 + \frac{1}{2\delta}\int_\Omega m_3^2$$

We will investigate the Landau-Lifshitz-Gilbert dynamics associated with this micromagnetic energy. Note that we no longer require m to be independent of z. The same analysis may be carried out for the full micromagnetic energy (3) in the regime under consideration.

It is well-known that the Landau-Lifshitz-Gilbert equation (2) can be written in the equivalent form

$$\frac{\partial m_\epsilon}{\partial t} + \alpha m_\epsilon \times \frac{\partial m_\epsilon}{\partial t} = (1+\alpha^2) m_\epsilon \times H(m_\epsilon); \qquad (12)$$

this version is more convenient for our analysis. The boundary and initial conditions are

$$\begin{aligned} \tfrac{\partial m_\epsilon}{\partial n} &= 0 \text{ at } \partial\Omega \\ m_\epsilon(0, x) &= g_\epsilon(x) \end{aligned} \qquad (13)$$

Here $\Omega = \omega \times (0, 1)$, $m_\epsilon : \Omega \to S^2$ and $g_\epsilon(x)$ is a suitable initial condition. The effective field $H(m_\epsilon) = -\delta E/\delta m_\epsilon$ is

$$H(m_\epsilon) = \Delta' m_\epsilon + \frac{1}{\epsilon^2}\frac{\partial^2 m_\epsilon}{\partial z^2} - \frac{1}{\delta} P_3 m_\epsilon$$

where $P_3 m_\epsilon = (m_\epsilon)_3 e_3$. We are interested in passing to the limit as $\epsilon, \delta \to 0$.

Theorem 1 *Consider the solution m_ϵ of (12)-(13), with initial data $g_\epsilon \in H^1(\Omega, S^2)$, such that $E_\epsilon(g_\epsilon) \leq C$ and $g_\epsilon \to g$ weakly in $(H^1(\Omega))^3$. As ϵ and δ tend to 0, there exists a sequence (also denoted $\{m_\epsilon\}$ for simplicity) and a limit m such that $m_\epsilon \to m$ weak* in*

$L^\infty(R^+; H^1(\Omega; S^2))$ *and strongly in* $(L^2((0,T)\times\Omega))^3$. *Moreover* $m = m(x,y)$, $m_3 = 0$ *and* m *satisfies*

$$\frac{\partial m}{\partial t} = -(\alpha + \frac{1}{\alpha}) m \times (m \times \Delta m) \quad \text{in } \omega \tag{14}$$

with boundary and initial conditions

$$\begin{aligned} &\tfrac{\partial m}{\partial n} = 0 \quad \text{on } \partial\omega \\ &m(0,x) = g(x). \end{aligned} \tag{15}$$

Proof. By the results of Alouges and Soyer [3] (see also Carbou [4]) we have the existence of weak solutions satisfying

$$E_\epsilon(t) + \int_0^t \int_\Omega |\frac{\partial m_\epsilon}{\partial t}|^2 \le E_\epsilon(0)$$

where

$$E_\epsilon(t) = \int_\Omega |\nabla' m_\epsilon|^2(t) + \frac{1}{\epsilon^2}\int_\Omega |\frac{\partial m_\epsilon}{\partial z}|^2(t) + \frac{1}{\delta}\int_\Omega |P_3 m_\epsilon|^2(t).$$

Since $E_\epsilon(0) \le C$ we obtain

$$\begin{aligned} \|\nabla m_\epsilon\|_{L^\infty(R^+, L^2(\Omega))} &\le C \\ \|\frac{\partial m_\epsilon}{\partial z}\|_{L^\infty(R^+, L^2(\Omega))} &\le C\epsilon \\ \|\frac{\partial m_\epsilon}{\partial t}\|_{L^2(R^+\times\Omega)} &\le C \\ \|P_3 m_\epsilon\|_{L^\infty(R^+, L^2(\Omega))} &\le C\sqrt{\delta}. \end{aligned}$$

Hence we have, for a subsequence,

$$\begin{aligned} &m_\epsilon \to^* m \quad \text{weakly in } L^\infty(R^+, (H^1(\Omega))^3) \\ &m_\epsilon \to m \quad \text{in } (L^2([0,T]\times\Omega))^3 \text{ for any } T > 0 \\ &\frac{\partial m_\epsilon}{\partial z} \to 0 \quad \text{in } L^\infty(R^+, (L^2(\Omega))^3) \\ &P_3 m_\epsilon \to 0 \quad \text{in } L^\infty(R^+, (L^2(\Omega))^3) \\ &\frac{\partial m_\epsilon}{\partial t} \to \frac{\partial m}{\partial t} \quad \text{weakly in } (L^2([0,T]\times\Omega))^3 \text{ for any } T > 0. \end{aligned}$$

It follows that

$$\begin{aligned} &m_\epsilon \to m \quad \text{in } (L^p([0,T]\times\Omega))^3 \text{ for any } p \ge 1 \text{ and any } T > 0 \\ &m_\epsilon \to m \quad \text{a.e } (x,t) \in R^+\times\Omega \quad \text{and } m_\epsilon(0) \to m(0) \text{ weakly in } (L^2(\Omega))^3 \\ &\frac{\partial m_{\epsilon,3}}{\partial t} \to 0 \quad \text{weakly in } L^2([0,T]\times\Omega) \text{ and any } T > 0 \\ &\nabla m_{\epsilon,3} \to 0 \quad \text{weakly in } L^2([0,T]\times\Omega) \text{ and any } T > 0. \end{aligned}$$

We conclude, in particular, that

$$|m| = 1, \quad m_3 = 0, \quad m = m(x,y), \quad m(0) = g.$$

Obviously, from energy estimate on g_ϵ and convergence of g_ϵ to g we have $g_3 = 0$, $g = g(x,y)$ and $|g| = 1$. From the above estimates and convergence results we see that there are two main difficulties in passing to the limit:

- the absence of a bound on $\frac{1}{\epsilon^2}\frac{\partial^2 m_\epsilon}{\partial z^2}$; and
- the absence of a bound on $\frac{1}{\delta}P_3 m_\epsilon$.

The first difficulty is relatively easy to deal with: in the following argument, all test functions ϕ will be independent of z. To overcome the second difficulty we will show that even without control of $\frac{m_{3,\epsilon}e_3}{\delta(\epsilon)}$, we still have a bound on $m_\epsilon \times \frac{m_{3,\epsilon}e_3}{\delta(\epsilon)}$; this permits passage to the limit in the equations. Then, using the structure of the equations we will can find an expression for the limit of $m_\epsilon \times \frac{m_{3,\epsilon}e_3}{\delta(\epsilon)}$ in terms of the limiting magnetization m.

Let's multiply the equations by $\phi \in (C^\infty(\omega_T))^3$ (here $\omega_T = \omega \times [0,T]$). Integrating over $Q_T = \Omega \times [0,T]$ we get

$$\int_{Q_T} \frac{\partial m_\epsilon}{\partial t}\cdot\phi + \alpha\int_{Q_T} m_\epsilon \times \frac{\partial m_\epsilon}{\partial t}\cdot\phi =$$

$$-(1+\alpha^2)[\Sigma_{i=1}^2 \int_{Q_T}(m_\epsilon \times \frac{\partial m_\epsilon}{\partial x_i})\cdot\frac{\partial\phi}{\partial x_i} + \frac{1}{\epsilon^2}(m_\epsilon \times \frac{\partial m_\epsilon}{\partial z})\cdot\frac{\partial \phi}{\partial z} + \frac{1}{\delta}m_\epsilon \times P_3 m_\epsilon\cdot\phi].$$

Taking the limit as $\epsilon \to 0$ we obtain

$$\int_{\omega_T}\frac{\partial m}{\partial t}\cdot\phi + \alpha\int_{\omega_T} m \times \frac{\partial m}{\partial t}\cdot\phi + (1+\alpha^2)\Sigma_{i=1}^2\int_{\omega_T}(m \times \frac{\partial m}{\partial x_i})\cdot\frac{\partial \phi}{\partial x_i} =$$

$$= -(1+\alpha^2)\lim_{\epsilon\to 0}\int_{Q_T}\frac{1}{\delta}m_\epsilon \times P_3 m_\epsilon \cdot\phi = \int_{\omega_T} A\cdot\phi$$

for some distribution A.

We see that for $A_\epsilon = \int_0^1 \frac{1}{\delta}m_\epsilon \times P_3 m_\epsilon$ we have $|\int_{\omega_T} A_\epsilon\cdot\phi| \le C\|\phi\|_{H^1(\omega_T)}$ for any $\epsilon > 0$ and any $\phi \in (C^\infty(\omega_T))^3$. Hence by the density of this space in $(H^1(\omega_T))^3$ we obtain $|\int_{\omega_T} A_\epsilon\cdot\phi| \le C\|\phi\|_{H^1(\omega_T)}$ for any $\epsilon > 0$ and any $\phi \in (H^1(\omega_T))^3$. Therefore $A_\epsilon \to A$ in the dual space of $H^1(\omega_T))^3$ and $\int_{\omega_T} A\cdot\phi$ makes sense for any $\phi \in (H^1(\omega_T))^3$.

Our plan is to show that $A = \beta m \times e_3$ for some scalar-valued β, then to evaluate β. Let us briefly explain why we must proceed this way. As $\epsilon \to 0$ we have no control over the term $\frac{1}{\delta}P_3 m_\epsilon$ (since our a priori estimates do not bound it in any space). As a consequence we are not allowed to say that $\frac{1}{\delta}m_\epsilon \times P_3 m_\epsilon \to m \times H$ for some H. However we may recover this statement by proving that $-(1+\alpha^2)\int_0^1 \frac{1}{\delta}m_\epsilon \times P_3 m_\epsilon dz \to A$ and $A = \beta m \times e_3$ for some β. This is what we are doing below.

First we show that A is orthogonal to e_3. Consider the choice $\phi = e_3\psi$ with $\psi \in C^\infty(\omega_T)$. It satisfies

$$\int_{Q_T} m_\epsilon \times P_3 m_\epsilon \cdot e_3\psi = 0 \text{ for all } \epsilon > 0,$$

so

$$0 = -(1+\alpha^2)\lim_{\epsilon\to 0}\int_{Q_T}\frac{1}{\delta}m_\epsilon \times P_3 m_\epsilon\cdot e_3\psi = \int_{\omega_T} A\cdot e_3\psi \text{ for all } \psi.$$

It follows that $A \cdot e_3 = 0$.

Now we show that A is orthogonal to m. Consider the choice $\phi = m\psi$ with $\psi \in C^\infty(\omega_T)$. This is admissible, since $m \in (H^1(\omega_T))^3$. Using test functions of this form, we easily obtain

$$\int_{\omega_T} A \cdot m\psi = 0 \text{ for any } \psi \in C^\infty(\omega_T).$$

Therefore A is orthogonal to m.

Since e_3, m and $m \times e_3$ form an orthonormal basis in R^3 at almost all points (x,t), we conclude that $A = \beta m \times e_3$. Let us find β. Since $m \times e_3\psi \in (H^1(\omega_T))^3$ we obtain

$$\int_{\omega_T} \frac{\partial m}{\partial t} \cdot (m \times e_3)\psi + \alpha \int_{\omega_T} m \times \frac{\partial m}{\partial t} \cdot (m \times e_3)\psi + (1+\alpha^2)\Sigma_{i=1}^2 \int_{\omega_T} (m \times \frac{\partial m}{\partial x_i}) \cdot \frac{\partial (m \times e_3)\psi}{\partial x_i}$$
$$= \int_{\omega_T} \beta\psi.$$

From this equation we obtain

$$\int_{\omega_T} (\frac{\partial m}{\partial t} \times m) \cdot e_3\psi = \int_{\omega_T} \beta\psi$$

and hence $\beta = (\frac{\partial m}{\partial t} \times m) \cdot e_3$.

Now we have

$$\int_{\omega_T} \frac{\partial m}{\partial t} \cdot \phi + \alpha \int_{\omega_T} m \times \frac{\partial m}{\partial t} \cdot \phi + (1+\alpha^2)\Sigma_{i=1}^2 \int_{\omega_T} (m \times \frac{\partial m}{\partial x_i}) \cdot \frac{\partial \phi}{\partial x_i} =$$
$$= \int_{\omega_T} [(\frac{\partial m}{\partial t} \times m) \cdot e_3](m \times e_3) \cdot \phi$$

and hence for any $\psi \in H^1(\omega_T)$

$$\int_{\omega_T} \frac{\partial m_1}{\partial t}\psi = \int_{\omega_T} m_2(\frac{\partial m_1}{\partial t}m_2 - \frac{\partial m_2}{\partial t}m_1)\psi$$
$$\int_{\omega_T} \frac{\partial m_2}{\partial t}\psi = -\int_{\omega_T} m_1(\frac{\partial m_1}{\partial t}m_2 - \frac{\partial m_2}{\partial t}m_1)\psi$$
$$\int_{\omega_T} \alpha(\frac{\partial m_1}{\partial t}m_2 - \frac{\partial m_2}{\partial t}m_1)\psi = -\int_{\omega_T} (1+\alpha^2)(m_1\Delta m_2 - m_2\Delta m_1)\psi$$

Taking $\psi = m_2\phi$, $\phi \in C^\infty(\omega_T)$ in the third equation, $\psi = \phi$ in first equation, we obtain

$$\int_{\omega_T} \frac{\partial m_1}{\partial t}\phi = -(\alpha + \frac{1}{\alpha}) \int_{\omega_T} m_2(m_1\Delta m_2 - m_2\Delta m_1)\phi.$$

On the other hand, for any $\phi \in C^\infty(\omega_T)$ we may take $\psi = m_1\phi$ in the third equation and $\psi = \phi$ in the second equation; this gives

$$\int_{\omega_T} \frac{\partial m_2}{\partial t}\phi = (\alpha + \frac{1}{\alpha}) \int_{\omega_T} m_1(m_1\Delta m_2 - m_2\Delta m_1)\phi.$$

It follows that

$$\frac{\partial m}{\partial t} = -(\alpha + \frac{1}{\alpha})m \times (m \times \Delta m)$$

in the sense of distributions. This completes the proof.

4 Conclusion

We have studied a simplified model of micromagnetic dynamics, which captures the essential physics of a ferromagnetic thin film in the regime $\epsilon|\log \epsilon| \ll d^2 \ll 1$. We proved that for this model, the gyromagnetic term of the 3D Landau-Lifshitz-Gilbert equation acts asymptotically as an additional damping term for the in-plane components of the magnetization. The effective equation is overdamped, with viscosity coefficient $\alpha + (1/\alpha)$. In particular, if $\alpha < 1$ then a decrease of the 3D damping actually increases the 2D effective damping.

References

[1] A. Aharoni, *Introduction to the Theory of Ferromagnetism*, Oxford Univ. Press, 1996

[2] H. Ammari, L. Halpern, K. Hamdache, *Asymptotic behavior of thin ferromagnetic films*, Asymptotic Analysis 24 (2000) 277-294

[3] F. Alouges, A. Soyeur, *On global weak solutions for Landau-Lifshitz equations: existence and non-uniqueness*, Nonlinear Anal. Theory Methods Appl. 18 (1992) 1071-1084

[4] G. Carbou, *Thin layers in micromagnetism*, Math. Models and Methods in Appl. Sciences 11, N9 (2001) 1529-1546.

[5] A. DeSimone, R.V. Kohn, S. Muller, F. Otto, *A reduced theory for thin film micromagnetics*, CPAM, LV 1408-1460

[6] C.J. Garcia-Cervera, *Magnetic domains and magnetic domain walls*, Ph.D. Thesis, New York University

[7] C.J. Garcia-Cervera, W. E, *Effective dynamics for ferromagnetic thin films*, J. Appl. Phys. 90, N1 (2001) 370-374

[8] G. Gioia, R.D. James, *Micromagnetics of very thin films*, Proc. Roy. Soc. London A 453 (1997) 213-223

[9] A. Hubert and R. Schäfer, *Magnetic Domains: The Analysis of Magnetic Microstructures*, Springer-Verlag, 1998

[10] R. Jerrard, H.M. Soner, *Dynamics of Ginzburg-Landau vortices*, ARMA 142, N2 (1998) 99-125

[11] R.V. Kohn, V.V. Slastikov, *Another thin-film limit of micromagnetics*, in preparation

[12] F.H. Lin, *Complex Ginzburg-Landau equations and dynamics of the vortices, filaments, and codimension-*2 *submanifolds*, Comm. Pure Appl. Math. 51, N4 (1998), 385-441

[13] R. Moser, in preparation.

[14] E. Sandier, S. Serfaty, *Limiting vortices for the Ginzburg-Landau equations*, Duke Math. J. 117, N3 (2003) 403-446