A few of Louis Nirenberg's many contributions to the theory of partial differential equations

Robert V. Kohn

1 Introduction

Mathematics is the language of science, and partial differential equations are a crucial component: they provide the language we use to describe—and the tools we use to understand—phenomena in many areas including geometry, engineering, and physics.

Louis Nirenberg's contributions to this field have been hugely influential. His impact includes the solution of many important problems, and—more importantly—the introduction of many fundamentally new ideas.

The depth, variety, and extent of his work make it difficult to synthesize. That challenge has nevertheless been undertaken twice, by YanYan Li [50] and by Tristan Rivière [73], with admirable success. Rather than attempt another synthesis, I shall focus here on six specific topics:

- his early work on the Weyl and Minkowski problems;
- his results with Shmuel Agmon and Avron Douglis on elliptic regularity;
- his paper with Fritz John on functions with bounded mean oscillation;
- his work with Luis Caffarelli and me on the Navier–Stokes equations;
- his results with Haim Brezis on nonlinear elliptic equations with critical exponents; and
- his work with Basilis Gidas, Wei-Ming Ni, and Henri Berestycki on the "method of moving planes" and the "sliding method."

My goal is to capture—to the extent possible in a few pages—the character of these contributions. I shall point to some related and/or subsequent work; however my

R.V. Kohn
Courant Institute of Mathematical Sciences, New York University, USA, e-mail: kohn@cims.nyu.edu

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discussions are necessarily incomplete, since a comprehensive review of even one of these topics would be a gargantuan task.

In focusing on these topics, I am necessarily omitting many important accomplishments; fortunately quite a few have been summarized elsewhere. For example, I do not touch his work at the interface between PDE and several complex variables—but these have been discussed by Joseph Kohn[1] and by Simon Donaldson[2]. The articles just cited also discuss other aspects of his work, and the surveys [50, 73] touch almost everything. Another rich source is [80], where leading researchers discuss five of his themes in the context of recent, related work of their own.

Louis is a friend, colleague, and role model to an entire community of mathematicians (myself included). A thoughtful and dedicated mentor, he has advised 46 PhD students (starting with Walter Littman in 1956, and ending with Kanishka Perera in 1997, according to the Mathematical Genealogy website), while also having a formative influence on countless postdocs and collaborators. His influence has been amplified by Louis' outstanding ability as an expositor: he writes in a way that invites the reader's participation, with detailed introductions that put his work in context and explain its main ideas. In addition to many research articles he has also written influential survey articles, including one on elliptic theory [69] and another on variational & topological methods [70]. His book *Topics in Nonlinear Functional Analysis*, written in 1974 and reprinted in 2001 [71], is still widely used today.

A cross-cutting theme in Louis' research is his exquisite taste in problems. One very successful mode has been to recognize, through specific challenges, the need for new PDE tools or estimates. His uncanny ability to identify such challenges—and to find the required tools or estimates—has been a major driver of his impact. The early work on the Weyl and Minkowski problems (Section 2) and the work with Brezis on nonlinear elliptic equations with critical exponents (Section 6) are examples of such work; additional examples include his paper with Newlander on the integrability of almost-complex structures [65] and his introduction (with Joseph Kohn) of the class of pseudodifferential operators [45].

A very different, also very successful mode has been to identify tools that are clearly important, and explore their scope systematically. His work with Agmon and Douglis on elliptic regularity (Section 3) and that with Gidas and Ni on the method of moving planes (Section 7) have this character. Another favorite example is his systematic treatment of interpolation inequalities (known as Gagliardo–Nirenberg inequalities, since they were found independently by E. Gagliardo [33] and by Nirenberg, who announced them at the 1958 International Congress of Mathematicians and published them as Section 2 of [69]).

But this tidy framework is too narrow to accommodate all Louis' work. In particular, he has always loved puzzles—especially ones involving estimates or inequalities—and this has led to many successful collaborations. The work with

[1] Louis Nirenberg receives the National Medal of Science, *Notices Amer. Math. Soc.* **43**(10), 1111–1116 (1996) (includes "Nirenberg's work in partial differential equations" by L. Caffarelli, and "Nirenberg's work in complex analysis" by J. J. Kohn).

[2] Donaldson, S.: On the work of Louis Nirenberg. *Notices Amer. Math. Soc.* **58**(3), 469–472 (2011).

Fritz John on functions of bounded mean oscillation (Section 4) is, in my view, an example of that type.

Louis' vision, leadership, and accomplishments have been recognized by many awards over the years; being selective, the list includes (besides the 2015 Abel Prize) the 1959 Bôcher prize, the 1982 Crafoord Prize, the 1994 Leroy P. Steele Prize for Lifetime Achievement, the 1995 National Medal of Science, and the 2010 Chern Medal.

His stature has led to many interviews[3] as well as video available at the Simons Foundation's Science Lives site[4]. These delightful resources capture (among other things) Louis' engaging wit, generosity, and taste.

2 The Weyl problem, the Minkowski problem, and fully nonlinear PDE in two space dimensions

Nirenberg's PhD thesis, completed in 1949, was entitled *The determination of a closed convex surface having given line element* [66]. The corresponding papers, published in 1953, are entitled *The Weyl and Minkowski problems in differential geometry in the large* and *On nonlinear elliptic partial differential equations and Hölder continuity* [67, 68]. This work proved two long-standing conjectures in differential geometry, and fundamentally advanced our understanding of fully-nonlinear PDE in two space dimensions. Not many PhD theses achieve so much!

The environment in which he did this work was rather unusual. The research group established at New York University by Richard Courant was still very small; its leaders (besides Courant) were Kurt Friedrichs, James J. Stoker and Fritz John (who arrived in 1946, shortly after Nirenberg's arrival as a graduate student). Government funding permitted substantial expansion after the war, and Courant and his colleagues had a remarkable eye for talent. As a result, Nirenberg's fellow PhD students were a truly remarkable group—including Avron Douglis, Harold Grad, Eugene Isaacson, Joseph Keller, Martin Kruskal, Peter Lax, and Cathleen Morawetz. (It was also a relatively large group: according to the Mathematical Genealogy website, NYU granted 37 mathematics PhD's in the four-year period 1948–1951.)

His thesis work provides an outstanding example of how specific challenges can lead to the development of fundamentally new tools. The challenges, in this case, were the Weyl and Minkowski problems—two easy-to-believe conjectures about two-dimensional surfaces in three-dimensional space, which had been open for many years. A framework for viewing them as nonlinear PDE problems was already well-established, and Hans Lewy had used it to obtain solutions when the data are

[3] Interview with Louis Nirenberg, interviewed by A. Jackson. *Notices Amer. Math. Soc.* **49**(4), 441–449 (2002), and Interview with Louis Nirenberg, interviewed by M. Raussen and C. Skau. *Newsletter of the European Mathematical Society*, Dec 2015, 33–38; reprinted in *Notices Amer. Math. Soc.* **63**(2), 135–140 (2016).

[4] Louis Nirenberg, interviewed by Jalal Shatah, on the Simons Foundation's Science Lives website https://www.simonsfoundation.org/2014/04/21/louis-nirenberg/. (Accessed 8 March 2018).

analytic [48, 49]. But the analytic category is very rigid! Each problem's natural formulation involves data that are a few times differentiable. Solving the problems in that setting required a new *a priori* estimate for fully-nonlinear PDE in two space dimensions. Nirenberg's fundamental contribution was to obtain that estimate.

The crucial estimate says that if u solves a PDE of the form $F(D^2u, Du, u, x) = 0$ in a two-dimensional domain,

(i) u, Du, and D^2u are continuous, with L^∞ norm at most K, and
(ii) the equation is elliptic with a positive ellipticity bound λ,

then in any subdomain D^2u is actually *Hölder continuous* (with a uniform bound depending only on K, λ, the C^1 norm of F, and the choice of subdomain). The key point, of course, is that while D^2u was only assumed to be bounded and continuous, the PDE assures that it is significantly better: Hölder continuous. Higher regularity follows by differentiating the equation and using linear PDE estimates (provided the regularity of F permits). Nirenberg's proof of this regularity theorem was related to the theory of quasiconformal mappings, drawing inspiration from Morrey's proof that 2D quasiconconformal mappings with bounded distortion are Hölder continuous [57].

As noted above, the specific challenges that led Nirenberg to consider this regularity issue were questions from differential geometry, raised by Weyl in 1916 and Minkowski in 1903. The Weyl problem has its roots in the fact that a convex surface in $\mathbb{R}^3$ has nonnegative Gaussian curvature. It seeks a sort of converse:

> Given a Riemannian metric g on the two-dimensional sphere S^2 with positive Gaussian curvature, can it be realized by a convex two-dimensional surface in $\mathbb{R}^3$? In other words, is there a map $H: S^2 \to \mathbb{R}^3$ such that $\|DH(x)v\|^2_{\mathbb{R}^3} = \|v\|^2_{g(x)}$ for every $x \in S^2$ and every $v \in T_xS^2$?

The Minkowski problem has its roots in the fact that if M is a strictly convex surface in $\mathbb{R}^3$, K_M is its Gaussian curvature, and $\nu_M: M \to S^2$ is its Gauss map (taking $x \in M$ to the outward unit normal to M at x), then (by elementary arguments) one has $\int_{S^2} \frac{x}{K_M(\nu_M^{-1}(x))}\, dA = 0$, where the variable of integration is $x = (x_1, x_2, x_3) \in S^2 \subset \mathbb{R}^3$ and the integral is with respect to surface area on S^2. The Minkowski problem seeks a sort of converse:

> Given a positive function K on S^2 satisfying $\int_{S^2} \frac{x}{K(x)}\, dA = 0$, is there a strictly convex surface M such that $K(x) = K_M(\nu_M^{-1}(x))$?

The suggestion to look at these problems came from James Stoker. This is not surprising in view of Stoker's longstanding interest in differential geometry (in fact, Stoker gave a new, simple proof in 1950 that a solution of the Minkowski problem is necessarily unique [81]). However Nirenberg has said that as a PhD student he worked most closely with Kurt Friedrichs[3].

Nirenberg's solution of each problem used what was known even then as "the method of continuity." Focusing (for simplicity of language) on the Weyl problem, the method consists of

(i) showing that the given metric (call it g_1) can be joined to the standard metric (call it g_0) by continuous path in the space of matrics with positive curvature (call it g_t, $0 \leq t \leq 1$);
(ii) showing that the set of t for which g_t is realizable is an open subset of $[0,1]$; and
(iii) showing that the set of t for which g_t is realizable is a closed subset of $[0,1]$.

The essence of this program was already present in Weyl's work; in fact, his 1916 paper [84] identified the fundamental issues and obtained several key estimates, though he lacked the PDE tools to complete the program. Nirenberg's treatment of (i) followed Weyl's. The proof of (ii) required solving a degenerate system of PDE's; Nirenberg's treatment used an iteration scheme, whose convergence was proved using estimates for certain 2nd order linear PDE (this was in large part a modern implementation of Weyl's ideas). Weyl had reduced the proof of (iii) to the study of a fully nonlinear PDE in two space dimensions, and he had shown that the solution was C^2, but this was not enough to conclude the argument. Nirenberg's regularity result—showing that the solution was actually $C^{2,\alpha}$ for some α—was the crucial ingredient permitting completion of the program.

His solution of the Minkowski problem followed a similar strategy. There, too, the argument used the method of continuity, and relied on prior work (in this case a 1938 paper by Lewy [49] and a 1939 paper by Miranda [56]) for identification of a suitable PDE-based framework. The prior work had reduced the analogue of (iii) to the study of a fully nonlinear PDE in two space dimensions, and Miranda had shown that the solution was C^2. Nirenberg's regularity result (showing that the solution was actually $C^{2,\alpha}$) was again the crucial ingredient permitting completion of the program.

In 1949—the year Nirenberg completed his PhD—another solution of the Weyl problem was published by the Soviet mathematician A.V. Pogorelov, using methods completely different from Nirenberg's. (Briefly: A.D. Alexandroff had shown the existence of a sort of weak solution, obtained by taking a limit of polyhedra; Pogorelov proved the regularity of those weak solutions.) Pogorelov also published a solution of the Minkowski problem in 1952. A discussion of Pogorelov's work and its relation to Nirenberg's can be found in the Math Reviews entry for [67], which is MR0058265. Pogorelov too was an outstanding mathematician, who did this work at the very beginning of his career. The independent solutions by Nirenberg and Pogorelov provide a reminder that while Soviet mathematics was remarkably strong in the post World War II period, communication with the West was quite limited.

In attacking the Weyl and Minkowski problems, Nirenberg was solving problems that others had claimed before. Indeed, a 1940 paper by Caccioppoli addressed the Weyl problem using the method of continuity. However, as Nirenberg wrote, in establishing point (iii) Cacciopoli "refers to previous publications on nonlinear second order elliptic equations (see [18] for references). These papers contain only sketches of proofs—details are not presented—and it is not clear that all the results mentioned there are fully established." Concerning the Minkowski problem: Miranda's 1939 paper [56] claimed a full solution, but it relied on Cacciopoli's not-fully-established results. By the time Nirenberg and Pogorelov worked on these problems, there seems to have been a consensus that the previous "solutions" were incomplete.

Nirenberg's proof of $C^{2,\alpha}$ regularity for solutions of fully-nonlinear elliptic equations was limited to two space dimensions. This was sufficient for the Weyl and Minkowski problems, since they involve two-dimensional surfaces in $\mathbb{R}^3$. It is natural, however, to ask what happens in higher dimensions: is a C^2 solution of a uniformly elliptic, fully nonlinear equation $F(D^2u, Du, u, x) = 0$ necessarily $C^{2,\alpha}$ in space dimension $n \geq 3$? The answer is yes, but the proof requires methods entirely different from those of Nirenberg's 1953 paper. (I thank N. Nadirashvili for input on this topic.) Briefly: if u solves such an equation, then for any i the partial derivative $v = \partial u/\partial x_i$ is a *viscosity solution* of the linear elliptic PDE obtained by formally differentiating the original equation (see, e.g., Corollary 1.3.2 of [61]). Since the leading-order term of this equation has the form $\sum a_{kl} \frac{\partial^2 v}{\partial x_k \partial x_l}$ with $a_{kl}(x)$ continuous, the regularity theory for viscosity solutions of linear elliptic equations is applicable, and it shows that v is $C^{1,\alpha}$ for some $\alpha > 0$ [19]. Interestingly, if the condition $u \in C^2$ is replaced by $u \in C^{1,1}$ then the argument breaks and higher regularity becomes false: a recent paper by Nadirashvili, Tkachev, and Vlăduţ [60] identified a nonlinear elliptic PDE of the form $F(D^2u) = 0$ in $\mathbb{R}^5$ with an (explicit) viscosity solution of the form $u(x) = p(x)/|x|$, where p is a homogeneous polynomial in x of degree 3. Since u is homogeneous of degree 2, it is C^2 except at $x = 0$, with bounded but discontinuous second derivatives at the origin.

3 Elliptic regularity for boundary value problems: the Agmon–Douglis–Nirenberg estimates

The 1950's was a period of rapid development in our understanding of elliptic PDE, and Nirenberg was a major player. The following discussion will focus on linear PDE with variable coefficients, since this is the heart of the matter. It should be understood, however, that these results are also crucial for the study of nonlinear PDE (for example, permitting existence theorems to be proved using fixed-point theorems or iteration arguments).

As background: by 1950 there was a rather comprehensive understanding of second-order elliptic PDE for a scalar-valued unknown: a treatment involving L^2-type estimates using Hilbert space methods was presented, for example, in volume 2 of Courant and Hilbert's *Methoden der Mathematischen Physik* [24], and estimates involving Hölder norms were established by Schauder in 1934. However a similarly general understanding of higher-order equations and elliptic systems was not yet available. Progress in those directions began in the early 50's with work by Vishik [83], Browder [15], and Gårding [34] among others. Another key development was the work of Calderón and Zygmund on singular integral operators [22], which provided the crucial tools needed for L^p-type estimates.

Nirenberg's contributions in the 50's included the following key advances:

- His 1955 paper with Avron Douglis, *Interior estimates for elliptic systems of partial differential equations* [26], extended elliptic theory to a much more general class of systems than had been considered before, obtaining Schauder-type interior estimates involving Hölder norms. Roughly speaking, this work identified what it should *mean* for a system to be elliptic. An important feature of the definition is that the system need not be of the same order in each unknown.
- His 1959 paper with Shmuel Agmon and Avron Douglis, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I* [1], provided estimates up to the boundary, for any elliptic boundary value problem involving a scalar-valued unknown. This work was notable both for its scope and for its method. Concerning the scope: while previous work provided a full understanding of problems with Dirichlet-type boundary conditions, the 1959 paper achieved something similar for *any* boundary condition satisfying the "complementing condition." (Roughly speaking: these are boundary conditions for which, in the homogeneous constant-coefficient case for a half-space, separation of variables reveals that a solution which is periodic on the boundary must decay exponentially toward the interior of the domain.) Concerning the method: the paper's starting point was was a study of the constant coefficient case in a half-space, obtaining an explicit solution analogous to the Poisson kernel representation of a harmonic function. These representations were then used to obtain estimates for the solution (up to the boundary, even for PDE's with variable coefficients in domains with curved boundaries), by applying tools from potential theory and the then-recently-developed theory of singular integral operators. This produced both estimates of Schauder type (estimating Hölder-type norms of the solution in terms of those of the data) and also analogous estimates of L^p-type. Related estimates were obtained by Felix Browder, in work done independently around the same time [16].

This ground-breaking work was done during a period of dramatic progress, to which many others contributed. The introductions of Nirenberg's papers are notable not only for their transparent discussions of the papers' methods and achievements, but also for their richly detailed discussions of related work by others.

The 1955 paper dealt with systems but obtained only interior estimates. The 1959 paper dealt with boundary estimates but was restricted to scalar-valued unknowns. It was of course a natural idea to combine the papers' methods, to obtain estimates up to the boundary for elliptic systems with general boundary conditions. Such results were already within view by 1959: the Introduction of [1] says "In this paper we shall derive 'estimates near the boundary' for elliptic equations of arbitrary order under general boundary conditions, not merely Dirichlet boundary conditions. We have obtained these results for general elliptic systems, but for simplicity, we treat here in detail the theory of a single equation for one function. Systems will be treated in a forthcoming paper." It took a few years to wrap things up (which is not surprising, considering the generality of the outcome):

- Nirenberg's 1964 paper with Shmuel Agmon and Avron Douglis, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying gen-*

eral boundary conditions. II [2], provided Schauder-type and L^p estimates up to the boundary, for boundary value problems involving the full range of systems considered in [26]. As in [1], the boundary conditions considered are essentially the most general ones permitting such estimates. (An indirect characterization is that for the homogeneous constant-coefficient case in a half-space, a solution which is periodic on the boundary must decay exponentially toward the interior of the domain; a more algebraic characterization is included in the paper. Such boundary conditions are said to satisfy the "complementing condition.") Like the earlier work [1] on scalar-valued unknowns, the analysis combines a thorough understanding of half-space problems with tools from potential theory and singular integral operators. However the paper's focus on general systems made the analysis of the half-space problems quite different from what was done in [1].

It is interesting to compare the scientific style of Nirenberg's earlier work on the Weyl and Minkowski problems with that of the papers just discussed. One could say that the earlier work was problem-driven, while that with Agmon and Douglis was method-driven. Indeed, the starting point of the earlier work was to solve the Weyl and Minkowski problems, while that of the later was to identify the full power and scope of certain methods. And yet, upon reflection the contrast is not so sharp: once he saw that the key to the Weyl and Minkowski problems was a regularity theorem for a 2nd order, fully-nonlinear elliptic PDE in two space dimensions, Nirenberg proved a rather general result of this type—capturing the full power of his method—and explored additional applications, for example to the existence of solutions to quasilinear PDE [68]. As for the work with Agmon and Douglis: no specific challenge was needed, since by the 1950's the importance of a priori estimates for elliptic equations and systems was well-established.

The Agmon–Douglis–Nirenberg estimates helped establish a sound foundation for the theory of elliptic PDE. Since the strength of this work lies partly in its generality, no example can capture its full importance. Let me nevertheless mention a favorite example, namely the applicability of this theory to linear elasticity. In the early days our understanding of elastostatics relied heavily on Korn's inequality (whose early proofs for traction-type boundary conditions were complicated and relied heavily on the special form of the problem). From a modern perspective, Korn's inequality is not completely irrelevant—it assures us, for example, that solutions to traction problems are unique up to rigid motions. But as far as elliptic estimates are concerned, the equations of elastostatics are just another example of an elliptic system to which the Agmon–Douglis–Nirenberg theory applies.

I also have a favorite example concerning the importance of permitting elliptic systems to be of different orders in different unknowns: if Ω is a bounded domain in $\mathbb{R}^n$, consider the generalized Stokes system

$$-\Delta u_i + \nabla_i p = f_i, \quad \operatorname{div} u = g \quad \text{in } \Omega$$

with boundary condition

$$\sum_{j=1}^{n} e_{ij}(u)n_j - pn_i = h_i \quad \text{at } \partial\Omega$$

(using the notation $e_{ij}(u) = \frac{1}{2}[\nabla_i u_j + \nabla_j u_i]$). When $n = 2$ or 3 and $g = 0$, problems of this form arise both in elasticity (when the material is incompressible) and in fluid dynamics (Stokes flow). The system is second-order in u and first-order in p, but it meets the requirements of the Agmon–Douglis–Nirenberg theory.

4 Functions with Bounded Mean Oscillation

Research is unpredictable: tools and results developed in a particular context often have impact in other contexts, leading to entirely unanticipated consequences. The focus of this section—Nirenberg's 1961 paper *On functions of bounded mean oscillation* with Fritz John [43]—provides a fine example.

This paper addressed the question: suppose a function u has *bounded mean oscillation* on a cube $Q_0 \subset \mathbb{R}^n$, in the sense that its mean oscillation on sub-cubes is finite:

$$\sup_{Q \subset Q_0} \frac{1}{|Q|} \int_Q |u - u_Q|\, dx =: \|u\|_{\mathrm{BMO}(Q_0)} < \infty$$

(here Q ranges over cubes contained in Q_0, $|Q|$ is the volume of Q, and u_Q is the average of u on Q). Elementary examples (for example $\log|x|$) show that u need not be L^∞, but suggest that u can be large only on very small sets. The John–Nirenberg paper quantified this; its main result was that if $\|u\|_{\mathrm{BMO}(Q_0)} \leq K$ then

$$|\{x : |u - u_{Q_0}| > \sigma\}| \leq Be^{-b\sigma/K}|Q_0|$$

for some constants B and b depending only on the dimension n. This yields, by elementary arguments, control of various norms of $u - u_{Q_0}$; in particular

$$\frac{1}{|Q_0|} \int_{Q_0} e^{\beta K^{-1}|u - u_{Q_0}|}\, dx \leq C$$

for constants β and C depending only on n, and

$$\frac{1}{|Q_0|} \int |u - u_{Q_0}|^p\, dx \leq C_{p,n} K^p \tag{1}$$

for any $p < \infty$. Focusing on the latter: while $u - u_{Q_0}$ is not uniformly of order K, its L^p norms are controlled for any $p < \infty$ as if that were the case.

The immediate motivation came from Fritz John's work on elasticity [41]. In nonlinear elasticity the deformation of an elastic body $\Omega \subset \mathbb{R}^3$ is a map $f : \Omega \to \mathbb{R}^3$. Writing $Df(x) = R(x)E(x)$ where $R(x)$ is a rotation and $E(x) = \left[(Df(x))^T Df(x)\right]^{1/2}$, the nonlinear elastic energy controls the nonlinear strain $|E(x) - I|$ but not the local rotation $R(x)$. In linear elasticity, Korn's inequality provides L^2 control of the infinitesimal rotation in terms of the L^2 norm of the linear strain; John's goal was a fully nonlinear analogue of this result. He found a proof that if the nonlinear strain is uniformly small on a cube, then the BMO norm of Df is also small:

$$\|E(x)-I\|_{L^\infty(Q)} \leq \varepsilon \quad \text{implies that} \quad \|Df\|_{\mathrm{BMO}(Q)} \leq C\varepsilon \tag{2}$$

provided ε is sufficiently small. Since E stays close to I by hypothesis, this is really an estimate on the oscillation of $R(x)$. Knowing Nirenberg's analytical power—and his love of inequalities—John drew Nirenberg into exploring the implications of (2). This was the origin of the John–Nirenberg paper; note that (1) with $p=2$ shows that $R(x)$ stays close in L^2 to its average on Q, turning (2) into the nonlinear Korn-like inequality

$$\frac{1}{|Q|}\int_Q |Df-(Df)_Q|^2\,dx \leq C\sup_{x\in Q}|E(x)-I|^2. \tag{3}$$

It was clear from the start that their estimates on BMO functions would have implications far beyond elasticity. Indeed the John–Nirenberg paper includes, as an application, a new proof of a result due to M. Weiss and A. Zygmund (namely: if G is periodic and $G(x+h)+G(x-h)-2G(x)=O\left(h/|\log h|^\beta\right)$ for some $\beta>1/2$ then G is the indefinite integral of some function g belonging to every L^p). A more dramatic application was provided by Jürgen Moser in the very same issue of Comm. Pure Appl. Math.: he used the John–Nirenberg theory to prove a Harnack inequality for the solution of a divergence-form elliptic equation

$$\sum_{i,j=1}^n \partial_i(a_{ij}(x)\partial_j u)=0,$$

when the matrix-valued function $a_{ij}(x)$ is merely L^∞ and uniformly elliptic [59]. The Hölder regularity of such u was a landmark result proved in 1957 by Ennio De Giorgi [25] and John Nash [62]. Moser had given a third proof in 1960 [58]. Harnack's inequality implies Hölder continuity by an elementary argument (this is Section 5 of [59]), so Moser's application of the John–Nirenberg estimate provided a fourth proof of the celebrated De Giorgi–Nash–Moser regularity theorem.

A different perspective on BMO began to emerge in the mid-60's, when J. Peetre, S. Spanne, and E. Stein observed (independently) that while singular integral operators (such as Riesz transforms) are not bounded linear operators on $L^\infty(\mathbb{R}^n)$, they *are* bounded linear operators on $\mathrm{BMO}(\mathbb{R}^n)$, the space of functions on all $\mathbb{R}^n$ such that

$$\sup_{\text{cubes Q}} \frac{1}{|Q|}\int_Q |u-u_Q|\,dx =: \|u\|_{\mathrm{BMO}(\mathbb{R}^n)} < \infty.$$

This was the first indication that $\mathrm{BMO}(\mathbb{R}^n)$ deserved attention as a function space, and would be the "right" substitute for L^∞ in many results of harmonic analysis. The correctness of this viewpoint became clear in the early 70's, when C. Fefferman showed [29, 30] that

(a) $f \in \mathrm{BMO}(\mathbb{R}^n)$ exactly if $f = g_0 + \sum_{j=1}^n R_j g_j$ where $g_0, \ldots g_n$ are L^∞ and R_j is the jth Riesz transform (which acts in Fourier space as $\xi_j/|\xi|$); and
(b) $\mathrm{BMO}(\mathbb{R}^n)$ is dual (using the L^2 inner product) to the Hardy space $H^1(\mathbb{R}^n)$ (which consists, by definition, of functions in L^1 whose Riesz transforms are also in L^1).

Returning to elasticity, it is natural to ask: is the John–Nirenberg theory of any use for the analysis of nonlinear elastic boundary value problems? A 1972 paper by John provides an attractive answer, by proving the uniqueness of nonlinear elastic equilibria when the boundary displacement is fixed and only deformations with uniformly small strain are considered [42]. The idea is relatively simple: writing the deformation as $f(x) = x + u(x)$ and proceeding as one would for linear elasticity (with u as the elastic displacement), one needs to show that the higher-order terms neglected in the linear theory are truly unimportant. This is done using a consequence of the John–Nirenberg theory slightly different from those displayed above: if a function g has small BMO norm and average value 0 on a (nice enough) domain Ω then

$$\int_\Omega |g|^3 \, dx \le C \|g\|_{\mathrm{BMO}(\Omega)} \int |g|^2 \, dx.$$

In the proof of the uniqueness theorem, this is applied with $g = Df_1 - Df_2$ where f_1 and f_2 are two uniformly-small-strain elastic equilibria.

Fritz John's arguments required uniform bounds on the strain. This is a serious handicap, since one rarely knows in advance that the solution of a nonlinear elasticity problem has uniformly small strain. Forty years after the work of John and Nirenberg, the relationship between nonlinear strain and rotation was revisited by G. Friesecke, R.D. James, and S. Müller [31]. They improved (3) by showing that

$$\int_Q |Df - (Df)_Q|^2 \, dx \le C \int_Q |E(x) - I|^2 \, dx, \tag{4}$$

and used this estimate to explore the connection between 3D elasticity and various plate theories [31, 32].

I started by noting the unpredictability of research progress. In 1961 John and Nirenberg anticipated connections to elasticity (this was after all their starting point), and they also anticipated connections to analysis (this is clear from their new proof of the Weiss–Zygmund result). But they could not have anticipated the deep links to harmonic analysis that emerged a decade later, and I don't think they anticipated that (4) would be true without assuming uniformly small strain.

5 Partial regularity for the 3D Navier–Stokes equations

I had the privilege of collaborating with Louis Nirenberg and Luis Caffarelli around 1981 on partial regularity for the incompressible Navier–Stokes equations. I was a 2nd-year postdoc at Courant in 1980-81, and Luis had just joined the faculty. It was

Louis' suggestion that we look together at Vladimir Scheffer's work on Navier–Stokes [75, 76], which none of us had read before. The discussions that followed were an incredible learning experience! Their outcome was our paper *Partial Regularity of Suitable Weak Solutions of the Navier Stokes Equations* [20].

The incompressible Navier–Stokes equations describe the flow of a viscous, Newtonian fluid (such as water). Focusing for simplicity on the problem in all $\mathbb{R}^3$ with unit viscosity and no forcing, the equations say that the velocity u and pressure p solve the initial value problem

$$\begin{aligned} u_t + u \cdot \nabla u - \Delta u + \nabla p &= 0 \\ \nabla \cdot u &= 0 \\ u(x,0) &= u_0(x). \end{aligned} \tag{5}$$

For this to be adequate as a description of the fluid, there should be a unique solution of (5) for any (sufficiently smooth) initial data u_0 with suitable decay as $|x| \to \infty$. We still don't know whether this is true or not. Indeed: if u_0 is smooth enough (and decays at infinity) there is a unique classical solution for a while at least, but for large initial data we cannot rule out the development of singularities in finite time. The solution can be continued for all time as a Leray–Hopf weak solution, but we do not know that such weak solutions are unique. (Nonuniqueness of Leray–Hopf weak solutions seems a real possibility, in view of recent progress including [17, 38, 40].)

The program that Scheffer began in the late 70's seems natural in hindsight, but at the time it was revolutionary. There was by then a well-established literature on the partial regularity of minimizers for problems from geometry and the calculus of variations. It was Scheffer's idea to study the partial regularity of weak solutions to the Navier–Stokes equations using similar methods. His main result was that for a suitably constructed weak solution, the singular set has $5/3$-dimensional parabolic Hausdorff measure zero in space-time. Our paper [20] obtained a similar result with $5/3$ replaced by 1. The improved result places substantial restrictions upon the geometry of the singular set; for example, in an axisymmetric solution the only possible location of a singularity is on the axis. (The definition of parabolic Hausdorff measure is similar to that of ordinary Hausdorff measure, except that it uses coverings not by balls but rather by parabolic cylinders Q_r having radius r in space and extent r^2 in time.)

Can a solution with smooth initial data really develop a singularity? We still don't know. Leray suggested looking for self-similar singular solutions, i.e., ones of the form

$$u(x,t) = (T-t)^{-1/2} w\left(x/\sqrt{T-t}\right), \tag{6}$$

but we now know there are no such solutions with locally finite energy [63, 82]. Leray's ansatz can be generalized by looking for a solution that remains "bounded in similarity variables," i.e., such that

$$u(x,t) = (T-t)^{-1/2} w(y,s) \quad \text{where } y = x/\sqrt{T-t} \text{ and } s = -\ln(T-t).$$

This leads to an autonomous evolution for $w(y,s)$, namely

$$w_s + w \cdot \nabla w - \Delta w + \frac{1}{2} w + \frac{1}{2} y \cdot \nabla w + \nabla q = 0, \tag{7}$$

to be solved in all $\mathbb{R}^3$ and all sufficiently large s, with $\nabla \cdot w = 0$ and a suitable decay condition as $|y| \to \infty$. Leray's proposal was to look for a *stationary* solution of (7), but to give an example of a singular solution it would suffice to find *any* solution of (7) that exists for all $s > s_0$ and doesn't decay to 0 as $s \to \infty$. Alas, we have no idea whether such a w exists or not.

In looking for possible examples of singular solutions, it is natural to focus on solutions with special symmetry. Since the partial regularity theory does not rule out an axially symmetric solution developing a singularity along its axis, considerable attention has been devoted to the axially symmetric setting. The main result there is that if blowup occurs, then it must be "type II" in both space and time, in the sense that the functions $(T-t)^{1/2}|u(x,t)|$ and $(x_1^2+x_2^2)^{1/2}|u(x,t)|$ must both be unbounded as t approaches the singular time T. Paraphrasing the first of these estimates: in the axially symmetric setting (with symmetry around the x_3 axis), if a solution blows up at time T then its L^∞ norm must grow faster than $(T-t)^{-1/2}$, and the associated solution of (7) must have $\|w\|_{L^\infty} \to \infty$ as $s \to \infty$ [23, 44, 78].

Returning for a moment to Scheffer's program, it is natural to hope for a proof that the parabolic Hausdorff dimension of the singular set is strictly less than 1. Alas, it seems that this would require an entirely new approach. Indeed, Scheffer's results and ours rely mainly on a "generalized energy inequality" (equation (11) below). The generalized energy inequality permits a nonzero forcing term f on the right-hand side of the Navier–Stokes equation provided that $u \cdot f \leq 0$, and it permits u to be discontinuous in time provided that $|u|^2$ only jumps downward. Using observations such as these, Scheffer has shown that the generalized energy inequality is consistent with u being singular on a set of parabolic Hausdorff dimension α for any $\alpha < 1$ [77]. Thus, the result of [20] seems to be more or less optimal, if the generalized energy inequality is to be used as the main tool and parabolic Hausdorff measure is used to measure the size of the singular set. (There are other ways to measure the size of the singular set; for some results using "box-counting dimension" see [46] the references cited there.)

The rest of this section provides a little more detail concerning the contributions of [20]. The main ingredients of a partial regularity theorem are:

(a) a weak solution, with some global estimates;
(b) a result of the form "locally sufficiently small implies regular;" and
(c) a covering argument.

Concerning (a): multiplying the Navier–Stokes equation by u, integrating in space, and integrating by parts leads formally to $\frac{d}{dt}\int |u|^2\,dx + 2\int |\nabla u|^2\,dx = 0$. For Leray–Hopf weak solutions the formal argument breaks down but we have still have an energy inequality:

$$\int_{\mathbb{R}^3 \times \{t\}} |u|^2\,dx + 2\iint_{\mathbb{R}^3 \times (0,t)} |\nabla u|^2\,dx\,d\tau \leq \int_{\mathbb{R}^3} |u_0|^2\,dx \tag{8}$$

where u_0 is the initial data and we focus for simplicity only on the whole-space problem. This clearly implies

$$\int_{\mathbb{R}^3\times\{t\}} |u|^2\,dx \le M \quad \text{and} \quad \iint_{\mathbb{R}^3\times(0,t)} |\nabla u|^2\,dx\,dt \le M/2 \tag{9}$$

for all t, where $M = \int_{\mathbb{R}^3} |u_0|^2\,dx$ is fixed by the initial data. It also implies that

$$\iint_{\mathbb{R}^3\times(0,t)} |u|^{10/3} + |p|^{5/3}\,dx\,d\tau \le CM^{5/3} \tag{10}$$

for all t. (The estimate for u follows from (9) using the Gagliardo–Nirenberg estimate $\int_{\mathbb{R}^3} |u|^{10/3}\,dx \le C\left(\int_{\mathbb{R}^3} |\nabla u|^2\,dx\right)\left(\int_{\mathbb{R}^3} |u|^2\right)^{2/3}$ and integration in time. The estimate for p follows from that for u, since we are discussing the whole-space problem: taking the divergence of the equation gives $\Delta p = -\sum_{i,j=1}^3 \nabla_i\nabla_j(u_iu_j)$, and for each i,j the singular integral operator $\Delta^{-1}\nabla_i\nabla_j$ is a bounded linear map from $L^{5/3}$ to itself.)

The energy inequality (8) is global, but partial regularity is a local matter. Therefore we need something similar but more local—a *generalized energy inequality*. For a smooth, compactly supported, scalar-valued function $\phi(x,t)$, multiplying the Navier–Stokes equation by $u\phi$, integrating in space, and integrating by parts leads formally to $\frac{d}{dt}\int |u|^2\phi\,dx + 2\int |\nabla u|^2\phi\,dx = \int |u|^2(\phi_t + \Delta\phi) + (|u|^2 + 2p)u\cdot\nabla\phi\,dx$; for weak solutions the formal argument breaks down, but (for suitably-constructed weak solutions) one gets the *generalized energy inequality*

$$2\iint |\nabla u|^2\phi\,dx\,dt \le \iint |u|^2(\phi_t + \Delta\phi) + (|u|^2 + 2p)u\cdot\nabla\phi\,dx\,dt \tag{11}$$

for smooth, compactly-supported functions ϕ such that $\phi \ge 0$.

Concerning (b): The Navier–Stokes equation has the following scale invariance: if $u(x,t)$ and $p(x,t)$ solve (5) then so does

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \quad p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t) \tag{12}$$

for any $\lambda > 0$. A result of the form “locally small implies regular” should be scale-invariant; in other words its hypothesis should have “dimension zero” under the convention that each spatial dimension x_i has dimension 1, time t has dimension 2, each velocity component u_i has dimension -1, the pressure p has dimension -2, and $\partial/\partial x_i$ has dimension -1. Note that in this parabolic setting, a local estimate should involve an integral over a parabolic cylinder

$$Q_r(x_0,t_0) = \{(x,t) : |x - x_0| < r,\ t_0 - r^2 < t < t_0\}.$$

The heart of the partial regularity theory in [20] is the following “locally small implies regular” result: there is a constant $\varepsilon_0 > 0$ such that

$$\limsup_{r\to 0}\, r^{-1}\iint_{Q_r(x_0,t_0)} |\nabla u|^2\,dx\,dt < \varepsilon_0 \quad \text{implies that } u \text{ is regular at } (x_0,t_0). \tag{13}$$

The proof makes use of a rather different "locally small implies regular" result: there is a constant $\varepsilon_1 > 0$ such that

$$r^{-2}\iint_{Q_r(x_0,t_0)} |u|^3 + |p|^{3/2}\,dx\,dt < \varepsilon_1 \ \text{ implies that } u \text{ is regular on } Q_{r/2}(x_0,t_0). \tag{14}$$

(This is a simplified version of Proposition 1 and Corollary 1 of [20]. The result there was more complicated, because it was not known at the time that for the solution of Navier–Stokes in a bounded domain the pressure was in $L^{3/2}$.) The latter estimate (and its proof) are quite close to what Scheffer had done before.

I will not attempt to discuss the proofs of these results, except to remark upon the relation between them: the proof of (13) in [20] proceeds by showing, roughly speaking, that if $r^{-1}\iint_{Q_r(x_0,t_0)} |\nabla u|^2\,dx\,dt$ is small enough then $r^{-2}\iint_{Q_r(x_0,t_0)} |u|^3 + |p|^{3/2}\,dx\,dt$ decays as r decreases, becoming eventually less than ε_1. Alternative proofs of these "locally small implies regular" results have since been given by others [47, 53]. A well-organized and modern exposition is available in [74].

Concerning (c): the covering arguments used to estimate the size of the singular set are quite standard. Using (10) and (14) one can show that the singular set has $5/3$-dimensional parabolic Hausdorff measure zero. Indeed, by (14) and Hölder's inequality, if (x_0,t_0) is a singular point then for any $r > 0$ the parabolic cylinder Q_r centered at (x_0,t_0) has

$$r^{-5/3}\iint_{Q_r} |u|^{10/3} + |p|^{5/3}\,dx\,dt \ge \varepsilon_1'$$

for some fixed positive constant ε_1'. By a parabolic variant of the Vitali covering lemma, one concludes that for any $\delta > 0$ the singular set is contained in a union of parabolic cylinders Q_j whose radii $r_j < \delta$ satisfy

$$\sum r_i^{5/3} \le C \iint_{\cup_j Q_j} |u|^{10/3} + |p|^{5/3}\,dx\,dt.$$

As $\delta \to 0$ this shows that the singular set has Lebesgue measure 0; since $\cup_j Q_j$ is contained in a δ-neighborhood of the singular set, the right hand side of the preceding estimate tends to 0 as $\delta \to 0$. So the singular set has $5/3$-dimensional parabolic Hausdorff measure 0. (This argument is close to what Scheffer did in [75, 76].)

The proof that the singular set has one-dimensional parabolic measure zero proceeds similarly, except that it combines the small-implies-regular result (13) with the global estimate on $\iint |\nabla u|^2\,dx\,dt$. It estimates the one-dimensional measure whereas the previous argument estimated the $5/3$-dimensional measure, because it relies on a global estimate for $\iint |\nabla u|^2\,dx\,dt$ (which has scaling dimension 1) whereas the previous argument relied on a global estimate for $\iint |u|^{10/3} + |p|^{5/3}\,dx\,dt$ (which has scaling dimension $5/3$).

Evidently, the outcome of the argument requires a suitable synergy between the form of the small-implies-regular result and the global estimate being used. Our paper [20] obtained additional results by considering global energy-type estimates with weighted norms, associated with formal calculations of $\frac{d}{dt}\int |u|^2|x|\,dx + 2\int |\nabla u|^2|x|\,dx$ and $\frac{d}{dt}\int |u|^2|x|^{-1}\,dx + 2\int |\nabla u|^2|x|^{-1}\,dx$. In doing so, we needed some analogues of the Gagliardo–Nirenberg interpolation inequalities in norms weighted by powers of $|x|$. Convinced that such estimates would have other uses as well, we wrote a separate paper on this topic [21]. The estimates proved there have indeed been used in many settings, and they have been generalized in various ways—for example to interpolation estimates involving weighted Hölder norms [51] and fractional derivatives [64]. The extremals and sharp constants for these estimates have also attracted considerable attention (see, e.g., [27, 52]).

6 Nonlinear elliptic equations involving critical exponents

Nirenberg's 1983 paper with Haim Brezis, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents* [14], was a landmark development in our understanding of semilinear PDE involving critical exponents. Its focus was the existence of solutions to

$$\begin{aligned} -\Delta u &= u^p + f(x,u) && \text{in } \Omega \\ u &> 0 && \text{in } \Omega \\ u &= 0 && \text{at } \partial\Omega \end{aligned} \tag{15}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ when $n \geq 3$, p is the "critical exponent"

$$p = \frac{n+2}{n-2},$$

and $f(x,u)$ grows slower than u^p at infinity.

To explain the issues, it is convenient to focus on the special case $f(x,u) = a(x)u$, when the PDE becomes

$$-\Delta u = u^p + a(x)u. \tag{16}$$

A necessary condition for the existence of a positive solution is that

$$\text{the first Dirichlet eigenvalue of } -\Delta - a \text{ is positive,} \tag{17}$$

as one easily verifies by multiplying (16) by the associated eigenfunction and integrating by parts. This condition is definitely not sufficient, since by Pohozhaev's identity there is no solution when $a(x) = 0$ and Ω is star-shaped.

To understand why $p = (n+2)/(n-2)$ is special, note that $p = (n+2)/(n-2)$ is equivalent to $p+1 = 2n/(n-2)$, the exponent that appears in the scale-invariant Sobolev inequality

$$\|u\|_{L^{2n/(n-2)}(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)} \quad \text{for } u \in H_0^1(\Omega). \tag{18}$$

A key point is that bounded sequences in $H_0^1(\Omega)$ are precompact in L^{q+1} for $q < (n+2)/(n-2)$, but not in $L^{2n/(n-2)}$. This is relevant to the problem at hand because when the exponent is subcritical (i.e., when p is replaced by q such that $1 < q < (n+2)/(n-2)$) there are straightforward variational approaches, either

(i) seeking a positive critical point of

$$\int_\Omega \frac{1}{2}|\nabla u|^2 - \frac{1}{q+1}|u|^{q+1} - \frac{1}{2}a(x)u^2\,dx, \tag{19}$$

or else

(ii) solving the variational problem

$$\inf_{\substack{\int_\Omega |u|^{q+1}\,dx=1\\ u=0 \text{ at } \partial\Omega}} \int_\Omega |\nabla u|^2 - a(x)u^2\,dx \tag{20}$$

(for which the Euler–Lagrange equation is $-\Delta u = a(x)u + \mu u^q$ with μ constant; the eigenvalue condition (17) assures that $\mu > 0$, so that a well-chosen scalar multiple of u solves (16)).

When q is subcritical approach (i) is tractable since the functional (19) satisfies the Palais–Smale condition, and approach (ii) also works since the direct method of the calculus of variations applies straightforwardly to (20). For the critical exponent, however, neither approach works (at least, not straightforwardly): (i) is dubious since the analogue of (19) doesn't satisfy the Palais–Smale condition; and (ii) is dubious since the constraint $\int_\Omega |u|^{p+1}\,dx = 1$ is not preserved under weak convergence in $H_0^1(\Omega)$. Moreover this is not just a technical issue, since (as noted above) there is in fact no positive solution in a star-shaped domain when $p = (n+2)/(n-2)$ and $a(x) = 0$.

The essential phenomenon here is the study of a variational problem involving both $\int |\nabla u|^2\,dx$ and $\int |u|^{p+1}\,dx$, for the special value of p where the latter is controlled by the former but with a lack of compactness. What intrigued Brezis and Nirenberg was the observation that the existence or nonexistence of a solution can depend, for such problems, on the presence (and form of) "lower order terms" such as $\int a(x)u^2\,dx$. This observation had already been made in a special case by Thierry Aubin, in a 1976 paper on the Yamabe problem [4] (a problem from geometry which is easily reduced to solving a PDE quite similar to (16) but on a Riemannian manifold without boundary rather than a domain in $\mathbb{R}^n$). Indeed it was Aubin's work that attracted their attention to this area.

Actually, variational problems involving a lack of compactness arise in a great variety of settings. The work of P.L. Lions on concentration compactness [54, 55] is a rich source of examples; in a different direction, the recent paper [35] by Ghoussoub and Robert discusses many examples involving the existence of extremals for Sobolev-type inequalities involving weighted norms.

The goal of the Brezis–Nirenberg paper [14] was to understand when and how the presence of a "lower order term" permits the variational approaches (i) or (ii) to succeed even when the exponent is critical. Their results include a rather complete understanding about when approach (ii)—based on a minimization analogous to (20)—suffices to solve (16) in space dimension $n \geq 4$. Focusing on this part of the story for a moment, let

$$S = \inf_{\substack{\int_\Omega |u|^{p+1}\,dx=1 \\ u=0 \text{ at } \partial\Omega}} \int_\Omega |\nabla u|^2\,dx \tag{21}$$

with $p = (n+2)/(n-2)$. (Clearly $S^{-1/2}$ is the best constant for the scale-invariant Sobolev estimate (18); the value of this constant doesn't depend on Ω and is the same as the best constant for the analogous scale invariant inequality in all $\mathbb{R}^n$; in particular, the value of S is known.) Now let J be the minimum value of (20) with q replaced by p:

$$J = \inf_{\substack{\int_\Omega |w|^{p+1}\,dx=1 \\ w=0 \text{ at } \partial\Omega}} \int_\Omega |\nabla u|^2 - a(x)u^2\,dx. \tag{22}$$

Brezis and Nirenberg showed that when $n \geq 4$ and the eigenvalue condition (17) holds, the the following are equivalent:

(a) $a(x) > 0$ somewhere in Ω
(b) $J < S$
(c) the minimum defining J is achieved.

(This statement combines several of the results in [14], following the lead of [12].) The proofs that $(b) \to (c)$ and $(c) \to (a)$ are relatively elementary, and they work even when $n = 3$. The assertion $(a) \to (b)$—proved using a well-chosen test function for J—is what restricts the result to $n \geq 4$.

The case $n = 3$ is surprisingly different, and the treatment in [14] was limited to the case when Ω is a ball and $a(x)$ is constant. A full understanding was achieved only in 2002 by Druet [28]; the $n = 3$ analogue of assertion (a) turns out to be that $g(x,x) > 0$ somewhere in Ω, where $g(x,y)$ is the regular part of the Green's function for $-\Delta - a$ on Ω.

I have focused thus far on approach (ii), which minimizes a suitable functional subject to the constraint $\int_\Omega |u|^{p+1}\,dx = 1$. Brezis and Nirenberg also studied approach (i), which is more useful when $f(x,u)$ is nonlinear in u, for example when the PDE is

$$-\Delta u = u^p + \mu u^q \tag{23}$$

where $q < p = (n+2)/(n-2)$ and $\mu > 0$ is constant. If p were subcritical it would be standard to find a critical point using the mountain-pass lemma. When p is critical, they show this still works (despite the failure of the Palais–Smale condition) when the min-max value of the functional (the critical value, so to speak) is strictly less than $\frac{1}{n}S^{n/2}$. Using this result, they show (for example) that when $n \geq 4$ equation (23) has a positive solution for any $\mu > 0$ and any bounded Ω.

I have already mentioned Pierre-Louis Lions' work on concentration compactness, which was roughly contemporaneous with [14]. While its focus was very similar—namely, variational problems with a lack of compactness—its method was rather different. Briefly: Lions focused on classifying the mechanisms by which compactness can be lost (and developing methods for ruling them out in specific examples), while Brezis and Nirenberg focused more sharply on a particular class of problems. The two investigations complement each other nicely: conditions for existence analogous to $J < S$ show up also in Lions' work, but Brezis and Nirenberg achieved a more complete understanding for the particular problems they addressed.

We have thus far discussed two particular methods for finding solutions of (15). Their failure does not necessarily imply nonexistence, as [14] makes clear by pointing to examples (such as $-\Delta u = u^p$ in the shell $\{1 < |x| < 2\}$). The nonexistence theorems in [14] are mainly for star-shaped domains, proved using Pohozhaev's identity or something similar. The fact that $-\Delta u = u^p$ has a positive solution in a shell but not in a star-shaped domain suggests that the *topology* of Ω might be relevant—and this was confirmed by Bahri and Coron in a 1988 paper [5], which developed an approach to existence theorems that takes advantage of nontrivial topology.

7 The method of moving planes and the sliding method

Nirenberg's 1979 paper with Basilis Gidas and Wei-Ming Ni, *Symmetry and related properties via the maximum principle* [36], began the development of a powerful and flexible toolkit for showing that the solutions of certain nonlinear elliptic PDE respect the symmetry suggested by their boundary conditions. Their approach, which soon became known as the *method of moving planes*, drew inspiration from work by Alexandroff on problems from geometry (for which their citation was [39]) and work by Serrin on PDE's with overdetermined boundary conditions [79]. The essential contribution of [36] was to show that far from being a trick that solves a few specific problems, the method of moving planes provides an intuitive and flexible approach for proving the symmetry of positive solutions, for a broad class of nonlinear PDE. While the 1979 paper [36] focused mainly on problems in bounded domains, it also considered some problems in all $\mathbb{R}^n$, and a followup paper [37] obtained additional results in that setting.

The method of moving planes is particularly well-suited to the study of positive solutions of equations of the form $\Delta u + f(u) = 0$ (as I'll discuss in some detail below). The introduction of [36] points briefly to the equations $\Delta u + u^{(n+2)/(n-2)} = 0$ and $\Delta u - u^{(n+2)/(n-2)} = 0$ in space dimension $n > 2$ as motivating examples, explaining their relevance to Yang–Mills field theory and geometry. However the paper is method-oriented not application-oriented, written with confidence that the method of moving planes would in due course find many applications. And indeed it has! While a survey is beyond beyond the scope of this article (and beyond the expertise of this author), let me mention one recent thread. The methods of Gidas, Ni, and Nirenberg have been extended to positive solutions of some *nonlocal* problems, by considering equivalent local problems in one more space dimension; for discussion and selected references see the segment of [80] by Xavier Cabré.

In the late 80's Nirenberg returned to this area in a fruitful collaboration with Henri Berestycki. Their focus was on certain nonlinear PDE's in infinite cylinders, whose solutions describe moving combustion fronts. In this setting, a key goal is to prove monotonicity of the solution (with respect to the cylinder's axial variable). To achieve this goal, they introduced a maximum-principle-based approach to monotonicity, known as the "sliding method," whose spirit is similar to the method of moving planes [8, 9].

The early 90's saw another important development, of a methodological character. Since the method of moving planes and the sliding method rely on versions of the maximum principle, the early papers had to exercise considerable care to be sure the required versions of the maximum principle were true. Besides complicating the analysis, this limited the statements of the theorems, for example by not permitting domains with corners. However it was understood in the early 90's that a uniformly elliptic operator of the form $Lu = \sum a_{ij}(x)\partial_{ij}u + \sum b_i(x)\partial_i u + c(x)u$ with bounded, measurable coefficients satisfies a maximum principle in a domain Ω ($Lu \geq 0$ in Ω and $u \leq 0$ at $\partial\Omega$ implies $u \leq 0$ in Ω) provided only that Ω has sufficiently small measure. Nirenberg's 1991 paper with Berestycki, *On the method of moving planes and the sliding method* [10], shows how this version of the maximum principle permits dramatic simplification of the proofs of many results, and extends their validity to more general domains (e.g., ones with corners). (For an expository account of these developments with much more detail than given here, see [13].)

The preceding paragraphs are at best an incomplete survey of Nirenberg's work in this area. In his paper with Berestycki on problems in cylinders [9], a subtlety quite distinct from the sliding method involves understanding the solution's asymptotics at $\pm\infty$; this is analyzed using Nirenberg's 1963 results with Agmon [3] and related results by Pazy [72]. While [9] obtains qualitative results about solutions that are assumed to exist, a 1992 paper with Berestycki obtains rather complete information about the existence and uniqueness of traveling fronts for cylinder analogues of the most-studied one-dimensional models [11]. Later, Nirenberg wrote two papers with Berestycki and Caffarelli [6, 7] applying the method of moving planes or the sliding method to the monotonicity and symmetry of some problems in unbounded domains.

But achieving completeness is a hopeless task. Rather, let me try to communicate the elegant simplicity of the method of moving planes and the sliding method, by discussing two examples from Introduction of [10]. The first uses the method of moving planes:

> Let Ω be a bounded domain in $\mathbb{R}^n$ which is convex in the x_1 direction and symmetric about $x_1 = 0$. Suppose $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ solves

$$\begin{aligned} -\Delta u &= f(u) && \text{in } \Omega \\ u &> 0 && \text{in } \Omega \\ u &= 0 && \text{at } \partial\Omega \end{aligned} \tag{24}$$

where f is locally Lipschitz. Then u is symmetric with respect to x_1, and $\partial_1 u < 0$ for $x_1 > 0$.

The hypothesis that u be positive is crucial; for example, when Ω is a ball centered at 0 there are plenty of Dirichlet eigenfunctions that are not symmetric in x_1—but they are not positive. The hypothesis that Ω be convex is also crucial; for example, when Ω is the shell $\{1 < |x| < 2\}$ the equation $-\Delta u = u^q$ has a non-radial positive solution with $u = 0$ at $\partial\Omega$ when the space dimension is $n > 2$ and q is below but sufficiently close to the critical value $(n+2)/(n-2)$ [14].

The following argument is general, but we visualize it in Figure 1 by taking Ω to be a diamond. Writing $x = (x_1, y)$ for points in $\mathbb{R}^n$, let $-a = \inf_{x\in\Omega} x_1$. For $-a < \lambda < 0$ let T_λ be the hyperplane $x_1 = \lambda$, let Σ_λ be the part of Ω where x_1 is less than λ, and observe that the function $(x_1, y) \mapsto u(2\lambda - x_1, y)$ is the reflection of u about the hyperplane T_λ. The key idea is to compare u with its reflection, by considering the function

$$w_\lambda(x_1, y) = u(2\lambda - x_1, y) - u(x_1, y).$$

Since f is locally Lipschitz and u is bounded, w solves an equation of the form

$$-\Delta w_\lambda + c_\lambda(x) w_\lambda = 0 \tag{25}$$

in Σ_λ, with $c_\lambda(x)$ bounded. Moreover $w_\lambda \geq 0$ on $\partial\Sigma_\lambda$ (in fact, it vanishes on the part of the boundary where $x_1 = \lambda$ and it is strictly positive on the part of the boundary that belongs to $\partial\Omega$). The main task in the method of moving planes lies in proving that

$$w_\lambda > 0 \text{ on } \Sigma_\lambda \text{ whenever } -a < \lambda < 0. \tag{26}$$

The symmetry of u follows from (26) by elementary arguments combined with relatively standard applications of the maximum principle (for full details see [10] or [13]).

We now sketch the proof of (26), using the fact that the PDE (25) has a maximum principle on a domain of sufficiently small volume. When λ is close to $-a$ the set Σ_λ is thin in the x_1 direction; so the maximum principle applies and $w_\lambda > 0$ in Σ_λ. Now let $\mu \leq 0$ be the largest value such that $w_\lambda > 0$ on Σ_λ for $\lambda \in (-a, \mu)$. If $\mu = 0$ we're done, so we assume $\mu < 0$ and seek a contradiction. By continuity we have $w_\mu \geq 0$ in Σ_μ, and it follows (using a version of the usual maximum principle) that in fact $w_\mu > 0$ in Σ_μ. Now let K be a compact subset of Σ_μ such that $\Sigma_\mu \setminus K$ has small measure. Evidently w_μ is bounded away from 0 on K. Therefore (by continuity) $w_{\mu+\varepsilon}$ is strictly positive on K when ε is sufficiently small. Since $\Sigma_{\mu+\varepsilon} \setminus K$ has small volume, the maximum principle on sets with small volume shows that $w_{\mu+\varepsilon} > 0$ on $\Sigma_{\mu+\varepsilon}$. It follows that $w_{\mu+\varepsilon} > 0$ on the entire set $\Sigma_{\mu+\varepsilon}$, contradicting the definition of μ. Thus $\mu = 0$ and the argument is complete.

Turning to the sliding method: the following example is again from the Introduction of [10] (though the statement there is a bit more general).

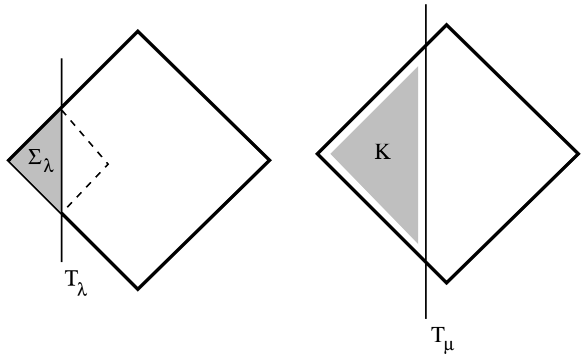

Fig. 1 The method of moving planes, when Ω is a diamond. Left: the hyperplane T_λ and the region Σ_λ (shaded); the broken line shows the boundary of the reflection of Σ_λ. Right: in the argument by contradiction, the set K (shaded) occupies most of Σ_μ.

Let Ω be a bounded domain in $\mathbb{R}^n$ which is convex in the x_1 direction, and assume $\partial\Omega$ contains no segment parallel to the x_1 axis. Suppose $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ solves

$$-\Delta u = f(u) \quad \text{in } \Omega \tag{27}$$

where f is locally Lipschitz, with boundary data such that

$$u = \phi \quad \text{at } \partial\Omega.$$

Assume that for any three points $x' = (x_1', y), x = (x_1, y), x'' = (x_1'', y)$ with $x', x'' \in \partial\Omega$, we have

$$\phi(x') \le u(x) \le \phi(x''). \tag{28}$$

Then u is strictly monotone in x_1, in the sense that

$$u(x_1 + \tau, y) > u(x_1, y) \text{ when } \tau > 0, \text{ if } (x_1, y) \text{ and } (x_1 + \tau, y) \text{ are both in } \Omega.$$

Furthermore, if f is differentiable then $\partial_1 u > 0$. Finally, u is the unique solution of the given boundary value problem satisfying (28).

The proof uses translation rather than reflection: for $\tau > 0$, it compares u with its translate by τ, by considering the difference

$$w_\tau(x_1, y) = u(x_1 + \tau, y) - u(x_1, y).$$

This function is defined in the domain D_τ obtained by intersecting Ω with its translation $\Omega - \tau e_1$. It once again solves a PDE of the form (25), and the (28) assures that $w_\tau \ge 0$ at ∂D_τ. The main task is to show that

$$w_\tau > 0 \text{ on } D_\tau \text{ for all } \tau > 0 \text{ such that } D_\tau \text{ is nonempty.} \tag{29}$$

The argument is parallel to that used in the first example. Briefly: when τ is large the domain D_τ is small and the maximum principle for (25) in small domains gives $w_\tau > 0$ on D_τ. On the other hand if $w_\tau > 0$ on D_τ for all $\tau > \tau_1 > 0$, an argument similar the one given before (relying once again on the maximum principle for domains with small volume) shows that that we also have $w_\tau > 0$ on D_τ for $\tau = \tau_1 - \varepsilon$ when ε is sufficiently small.

The monotonicity of u and the other conclusions follow from (29) by elementary arguments combined with relatively standard applications of the maximum principle (for full details see [10]).

8 Conclusion

As indicated by my title, I have discussed just a few of Louis Nirenberg's many contributions. The topics I have selected are important, but many topics I have omitted are also very important. Writing about—indeed, thinking about—Louis' impact is truly a humbling experience. It was a great pleasure to see his contributions recognized by the 2015 Abel Prize.

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