The Mathematics of Wrinkles and Folds

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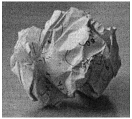
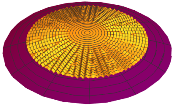
Orientation

The wrinkling and folding of thin elastic sheets is very familiar.
Today: a (mostly) variational perspective on how math can help.

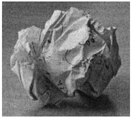
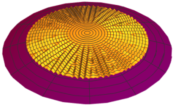




Physical context: rich phenomenology, readily observed; potential applications eg to metrology and device design; provocative analogies to other physical systems where defects and patterns form (eg liquid crystals, ferromagnets, martensitic phase transformation).

Mathematical context: elastic energy is very nonconvex, resembling a Landau theory from condensed matter physics; we see defects and patterns – but how to describe and analyze them? A developing chapter in the calc of varns: “energy-driven pattern formation.”

Orientation

Many MS18 minisymposia address aspects of this topic; see *Thin Structures: Defects, Patterns, & Bifurcations, I-IV* (Mon and Tues) and *Geometry & Elasticity, I-IV* (Wed and Thurs)

Annie Raoult's plenary talk *Models for thin prestrained structures* (Thurs pm) is also related.

Talk plan

(1) Getting started

The membrane and bending energies;
Mobius strips, d-cones, and more.

(2) Tensile wrinkling

The direction of wrinkling is clear;
but what determines the length scale?

(3) Compressive wrinkling

The direction of wrinkling is no longer clear; but energy scaling laws are still informative in some examples.

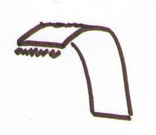
(4) Wrinkling and geometry

A current focus for several groups. Striking recent progress (but still many open questions).

A recurrent theme: energy scaling laws provide a convenient framework for analysis. They're obtained by combining a good ansatz (providing an upper bound) with an ansatz-free lower bound. In many settings this helps explain what we see, and why we see it.

Getting started

Paper is familiar, but already interesting.



Configuration of a sheet of paper is a map $g : \Omega \to \mathbb{R}^3$, where $\Omega \subset \mathbb{R}^2$ is the (undeformed) sheet's shape.

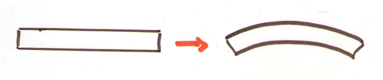
Elastic energy consists of membrane energy and bending energy (plus terms assoc to loads or bdry conds).

Membrane energy reflects in-plane stretching or compression of sheet; hence proportional to sheet thickness h. Integrand depends on "principal stretches" (eigenvalues of 2×2 matrix $(Dg^T Dg)^{1/2}$), derivable from 3D elastic law. Key feature: it prefers isometry. Simple example:

$$\text{Membrane energy} = h \int_\Omega \|(Dg^T Dg)^{1/2} - I\|^2 \, dx$$

Getting started

Bending energy reflects effect of curvature: if midplane is isometric, parallel surfaces won't be!



Lack of isometry grows linearly with distance to midplane. So

$$\text{Bending energy} = ch^3 \int_\Omega \kappa_1^2 + \kappa_2^2 \, dx$$

where κ_1, κ_2 are prin curvatures of (deformed) sheet.

The energy per unit thickness

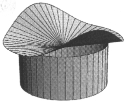
$$E_h = \int_\Omega \|(Dg^T Dg)^{1/2} - I\|^2 \, dx + ch^2 \int_\Omega \kappa_1^2 + \kappa_2^2 \, dx$$

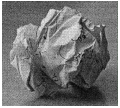
is like a Landau theory from condensed matter physics: a nonconvex term regularized by a higher-order singular perturbation.

After non-dimensionalization: for a sheet with extent L and thickness t, Ω is its normalized shape and h is the eccentricity t/L.

Getting started

Paper is already interesting.





When h is small, elastic energy vastly prefers isometry

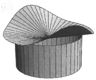
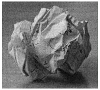
$$E_h = \int_\Omega \|(Dg^T Dg)^{1/2} - I\|^2 \, dx + ch^2 \int_\Omega \kappa_1^2 + \kappa_2^2 \, dx$$

Mobius band For some bc, isometry is consistent with finite bending energy. Then configuration minimizes bending subject to constraint of isometry, and min $E_h \sim h^2$.

d-cone, crumpling For other bc, there is no isometry with finite bending energy. Then the two terms are in conflict; this leads to defects (eg point singularities or folds). Signature of such defects: min $E_h \gg h^2$.

Example: for a conical configuration, curvature $\sim 1/r$ at dist r from center, leading to bending energy $h^2 \int r^{-2} \, r \, dr = \infty$. Smoothing near center costs membrane energy.

Some talks this week

- For thin ribbons (eg Mobius band), can energy minimization be reduced to a 1D var'l problem on the ribbon's midline? (Maria Giovanna Mora, MS24, Tues 8:30am)
- For bdry conditions that produce conical defects, can we understand the local structure & find the energy scaling law? (Heiner Olbermann, MS14, Mon 4:30pm)
- Crumpling is disordered, but ordered patterns can also achieve confinement. What can be done using relatively simple origami-based constructions? (Paul Plucinsky, MS24, Tues 9am)
- My models assume sheet is initially flat, elastically isotropic and homogeneous. What about sheets with prestrain, due eg to nonuniform growth or swelling? (Annie Raoult, IT9, Thurs 2pm)

Taking stock

Lessons thus far:

- Elastic energy of a thin sheet has the form

$$E_h = \text{membrane} + h^2 \text{ bending} + \text{terms assoc loads or bc}$$

- Scaling law wrt h provides a signature for the presence of defects (such as point defects or folds).
- Rich with challenges; even relatively simple problems (eg the mechanics of ribbons, and the local structure of point defects) still pose challenges.

Talk plan:

(1) Getting started

(2) Tensile wrinkling

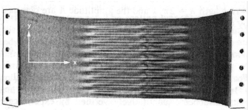
(3) Compressive wrinkling

(4) Wrinkling and geometry

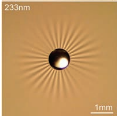
A thin sheet may wrinkle to avoid compression. When there is tension, the direction of the wrinkling is clear. But what sets the amplitude and length scale? (Note that they typically vary macroscopically.) Images from some experiments:

- hanging drapes (Vandeparre et al, PRL 2011)

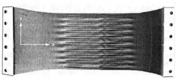
- stretched sheets (Cerda & Mahadevan, PRL 2003)

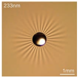


- water drop on floating sheet (Huang et al, Science 2007)



Framework for analysis of tensile wrinkling





Conjecture: wrinkled configurations resemble the ground state. So we focus on “energy scaling law,” i.e. how the min of

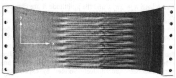
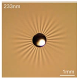
$$E_h = (\text{membrane energy}) \; + \; h^2(\text{bending energy}) + (\text{loads})$$

depends on h as $h \to 0$.

A key advantage: we lack a language to describe the pattern. The energy scaling law is amenable to rigorous analysis, and we learn a lot by identifying it.

When conjecture fails – when states seen in nature don’t resemble energy minimizers – that too is interesting.

Framework for analysis of tensile wrinkling

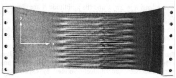
  

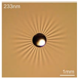
$$E_h = (\text{membrane energy}) \; + \; h^2(\text{bending energy}) + (\text{loads})$$

STEP 1: As $h \to 0$, energy min requires infinitesimal wrinkling. Analysis via relaxed variational problem, obtained by replacing membrane term by the elastic energy of an infinitesimally wrinkled sheet and dropping the bending term. (An old idea.)

Relaxed problem determines macroscopic features, including the tension, the wrinkled region, and "how much arclength must be wasted by wrinkling." Its min value is the limiting energy $\mathcal{E}_0 = \lim_{h \to 0} \min E_h$.

Framework for analysis of tensile wrinkling





$$E_h = (\text{membrane energy}) + h^2(\text{bending energy}) + (\text{loads})$$

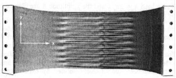
Step 2: When $h > 0$ scale of wrinkling must be positive, to keep bending energy finite. Wrinkling produces out-of-plane displacement, increasing membrane term. Local length scale of wrinkling is determined by competition between bending & membrane effects. (Recognized by Cerda & Mahadevan, 15 years ago).

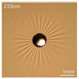
Let $\mathcal{E}_h = \min E_h$ be the min energy at fixed $h > 0$, and define

$$\mathcal{E}_h = \mathcal{E}_0 + \text{excess energy}.$$

An ansatz for the wrinkling gives an upper bound on the excess energy. Ansatz-free lower bounds assess the quality of upper bound. Since excess energy is linked to the length scale of wrinkling, such results – and their proofs – provide insight and intuition.

Framework for analysis of tensile wrinkling





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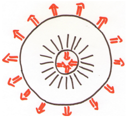
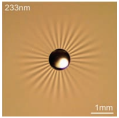
An example: the annulus problem

Annulus-shaped sheet, loaded by uniform tension at both boundaries. Captures essential physics of the “sheet-on-drop” experiment (Davidovitch et al, PNAS 2011).

No wrinkling at larger radii; lots of wrinkling at smaller radii, to avoid compression. Free boundary at $r = r_0$.





Energy scaling law studied with Peter Bella (CPAM 2014). Main conclusion: excess energy is linear in h,

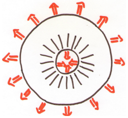
$$\mathcal{E}_0 + C_1 h \leq \min E_h \leq \mathcal{E}_0 + C_2 h$$

Really two assertions:

- upper bound (requires a good ansatz)
- lower bound (ansatz-free!)

An example: the annulus problem

Recall intuition: $h > 0$ forces finite-scale wrinkling, hence nontrivial out-of-plane displacement. Larger length scale reduces the bending energy but increases membrane term.

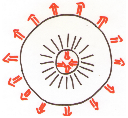


Let w be the out-of-plane displacement of the sheet.

Upper bound $\min E_h \leq \mathcal{E}_0 + C_2 h$ is not entirely trivial:

- Ansatz of form $w = f(r)\sin(\theta/h^{1/2})$ works near the center, but not near edge of wrinkled region. Its membrane term is too large there, giving excess energy of order $h|\log h|$.
- The log can be eliminated by being more careful:
  - (a) keep length scale of order $h^{1/2}$ near edge of wrinkled region, but let the profile of the wrinkling depend on r; or
  - (b) introduce a "cascade of wrinkles," changing the length scale near the edge of the wrinkled region (as seen in the hanging drape).
- The upper bound requires care near the edge of the wrinkled region. Is nature so careful? (Maybe not.)

An example: the annulus problem



Lower bound min $E_h \geq \mathcal{E}_0 + C_1 h$ has an interpolation inequality at its heart.

Argue by contradiction. Suppose a configuration exists with excess energy less than δh; show δ cannot be too small.

- Excess energy includes bending term, so out-of-plane displacement w has $h^2 \int |\nabla\nabla w|^2 \, dx \leq C\delta h$.
- Radial tension $\Rightarrow$ rays from origin are like stretched rubber bands; out-of-plane displacement makes them stretch more, increasing the membrane term. Thus out-of-plane displacement is controlled by excess energy: $\int w^2 \, dx \leq \delta h$.
- Inequality $\|\nabla w\|_{L^2} \leq C\|w\|_{L^2}^{1/2} \|\nabla\nabla w\|_{L^2}^{1/2}$ shows that $\|\nabla w\|_{L^2} \leq C\delta^{1/2}$.
- But: small slope $\Rightarrow$ no room to wrinkle $\Rightarrow$ configuration is essentially planar. This requires in-plane compression, hence large membrane energy – a contradiction.

A word about the hanging drape

Drape gathered at the top, hanging due to gravity. (Expt and ansatz-based analysis: Vandeparre et al, PRL 2011. Energy scaling law: Bella & Kohn, CPAM 2017.)

To save bending energy, wrinkles coarsen and/or sides spread. Scaling law of excess energy identifies # generations of coarsening, and whether spreading is significant.

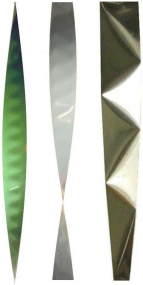
Coarsening of wrinkles cannot proceed too quickly, since it costs membrane energy. Analysis relies on a lemma estimating this cost.

An example without tension in the wrinkled region

Tension may control the direction of wrinkling even if there's no tension in the wrinkled region.

A recent example, with Ethan O'Brien: wrinkling in the center of a stretched, twisted ribbon (J Nonlin Sci 2018, building on modeling by Chopin et al J Elast 2015).

Expts (Chopin & Kudrolli, PRL 2013) show several regimes; my focus is on leftmost figure.



In regime of the leftmost figure: lines parallel to ribbon midline form helices. Arclength of helix is longest for lines at outer edge. In regime of interest, those are stretched while midline is in compression.

An example without tension in the wrinkled region

- Outer edges have little freedom, due to tension. As a result, relaxed problem predicts uniaxial compression in center, uniaxial tension near outer edges. Horizontal lines have strain 0.
- Wrinkling requires transverse deflection, costing excess membrane energy.
- Scaling of excess energy is $h^{4/3}$. (Different from annulus problem, since horizontal lines are not in tension.)

For more detail: go to Ethan O'Brien's talk in MS15 (Mon 4:30pm)

Taking stock

Lessons wrt tensile wrinkling:

- Relaxed problem predicts leading-order energy and macroscopic features, including extent of the wrinkled region and the uniaxial compression that is avoided by wrinkling.
- Scale of wrinkling is linked to scaling law of excess energy.
- Upper bound requires care at the edge of the wrinkled region (with features not typically seen in expts).
- Ansatz-free lower bound displays mechanism of competition between membrane and bending terms.

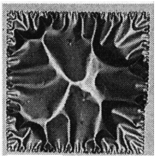
Talk plan:

(1) Getting started
(2) Tensile wrinkling
(3) Compressive wrinkling
(4) Wrinkling and geometry

Compressive wrinkling

In the absence of tension, the situation is different. One example:

Compression due to thermal mismatch:
a thin film bonded to a too-short bdry
(Lai et al, J Power Sources 2010)

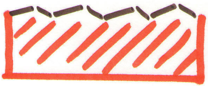


- Relaxed energy is 0, providing no information at all.
- Greater multiplicity of low-energy structures.
- Energy scaling law: $C_1 h \leq \min E_h \leq C_2 h$ (Ben Belgacem et al 2000, Jin & Sternberg 2001). Carries little info on bulk pattern, since energy of layer near bdry must have energy of order h.
- In general: in the absence of tension even the direction of wrinkling is unclear.

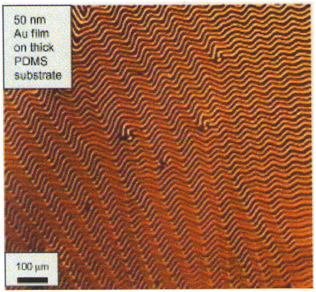
An example: the herringbone pattern

Work with Hoai-Minh Nguyen, concerning a thin, stiff layer bonded to a thick, compliant substrate (J Nonlin Sci 2013)

- stretch a polymer layer (biaxially)
- deposit the film
- release the polymer
- film buckles to avoid compression



Commonly seen: a herringbone pattern.

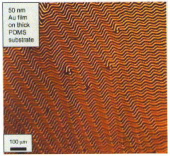


gold on pdms

Chen & Hutchinson, Scripta Mater 2004

An example: the herringbone pattern

We use a "small-slope" (von Karman) version of elasticity, writing (u_1, u_2, w) for the elastic displacement. Periodicity is assumed on some (large) scale L. The energy per unit area E_h has three terms:



(1) Membrane term captures fact that film's natural length is larger than that of the substrate:

$$\frac{h}{L^2}\int_{[0,L]^2} \left|e(u)+\tfrac{1}{2}\nabla w\otimes\nabla w-\eta I\right|^2 dx$$

(2) Bending term captures resistance to bending:

$$\frac{h^3}{L^2}\int_{[0,L]^2} |\nabla\nabla w|^2\, dx$$

(3) Substrate energy captures fact that substrate acts as a "spring", tending to keep film flat:

$$\frac{\alpha_s}{L^2}\left(\|w\|^2_{H^{1/2}}+\|u\|^2_{H^{1/2}}\right)$$

where $\|\phi\|^2_{H^{1/2}}=\sum|k||\hat{\phi}(k)|^2$

An example: the herringbone pattern

A brief explanation of the small-slope (von Karman) membrane energy

$$h \int |e(u) + \tfrac{1}{2}\nabla w \otimes \nabla w - \eta I|^2 \, dx \, dy$$

where (u_1, u_2, w) is the elastic displacement, and η is the misfit (both assumed small). Notation: $e(u)$ is the linear strain $\frac{1}{2}(Du + Du^T)$.

- 1D analogue: $\int |\partial_x u_1 + \frac{1}{2}(\partial_x w)^2 - \eta|^2 \, dx$
- Explanation: if $(x, 0) \mapsto (x + u_1(x), w(x))$ then local stretching is

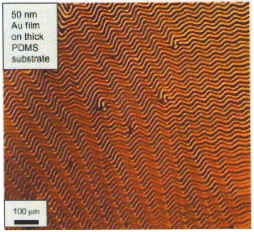
$$\sqrt{(1 + \partial_x u_1)^2 + (\partial_x w)^2} - 1 \approx \partial_x u_1 + \tfrac{1}{2}(\partial_x w)^2$$

An example: the herringbone pattern

$$E_h = \frac{h}{L^2}\int_{[0,L]^2} \left|e(u) + \tfrac{1}{2}\nabla w \otimes \nabla w - \eta I\right|^2 dx + \frac{h^3}{L^2}\int_{[0,L]^2} |\nabla\nabla w|^2\, dx + \frac{\alpha_s}{L^2}\left(\|w\|^2_{H^{1/2}} + \|u\|^2_{H^{1/2}}\right)$$

Summary of main results:

- energy of unbuckled state $w = u = 0$ is $\eta^2 h$;
- energy of herringbone pattern has order $\alpha_s^{2/3}\eta h$;
- $\min E_h \sim \min\{\eta^2 h, \alpha_s^{2/3}\eta h\}$, so herringbone achieves the energy scaling law when $\alpha_s^{2/3} \ll \eta$.

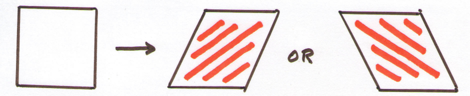


An example: the herringbone pattern

Essence of upper bound: good understanding of the pattern.

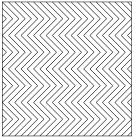
Film wants to expand (isotropically) relative to substrate.

- 1D wrinkling expands only transverse to the wrinkles
- a simple shear expands one diag dirn, compresses the other
- shear combined with wrinkling achieves isotropic expansion



Substrate prohibits large deformation; therefore the film mixes the two shear-combined-with-wrinkling variants. Thus, the herringbone pattern has two length scales:

- The smaller (the scale of the wrinkling) is set by competition between *bending* term and *substrate energy of w*.
- The larger one (scale of the phase mixture) must be s.t. the *substrate energy of u is insignificant*. (It is not fully determined.)



An example: the herringbone pattern

Lower bound has, at its heart, another interpoln ineq.

Claim: For any periodic (u_1, u_2, w), $E_h \geq C \min\{\eta^2 h, \alpha_s^{2/3} \eta h\}$. Take $L = 1$ for simplicity. Recall that

$$\text{membrane term} = h \int |e(u) + \tfrac{1}{2}\nabla w \otimes \nabla w - \eta I|^2 \, dx,$$

$$\text{bending term} = h^3 \|\nabla\nabla w\|_{L^2}^2, \quad \text{and} \quad \text{substrate term} \geq \alpha_s \|w\|_{H^{1/2}}^2.$$

CASE 1: If $\int |\nabla w|^2 \ll \eta$ then membrane $\gtrsim \eta^2 h$, since $e(u)$ has mean 0.

CASE 2: If $\int |\nabla w|^2 \gtrsim \eta$ use the interpolation inequality

$$\|\nabla w\|_{L^2} \lesssim \|\nabla\nabla w\|_{L^2}^{1/3} \|w\|_{H^{1/2}}^{2/3}$$

to see (using arith mean/geom mean ineq) that

$$\begin{aligned}
\text{Bending + substrate terms} &= h^3 \|\nabla\nabla w\|^2 + \tfrac{1}{2}\alpha_s \|w\|_{H^{1/2}}^2 + \tfrac{1}{2}\alpha_s \|w\|_{H^{1/2}}^2 \\
&\gtrsim \left(h^3 \|\nabla\nabla w\|^2 \alpha_s^2 \|w\|_{H^{1/2}}^4\right)^{1/3} \\
&\gtrsim h\alpha_s^{2/3} \|\nabla w\|_{L^2}^2 \gtrsim h\alpha_s^{2/3} \eta
\end{aligned}$$

An example: the herringbone pattern

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Taking stock

Lessons wrt compressive wrinkling:

- Relaxed problem is completely degenerate (no longer useful).
- We nevertheless understood the herringbone, as a mechanism for achieving the min energy scaling law.
- Uniqueness is not asserted! Indeed, less ordered patterns are also seen ("labyrinths"); do they achieve same scaling law?
- Lower bounds always rely on interpolation. How else to use knowledge that $\int |\nabla\nabla w|^2 \, dx$ scales like a neg power of h?

Talk plan:

(1) Getting started

(2) Tensile wrinkling

(3) Compressive wrinkling

(4) Wrinkling and geometry

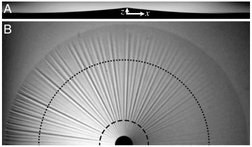
Geometry and wrinkling

The interaction of geometry and wrinkling is a current frontier. Too rich and too new for a synthesis; instead, three snapshots:

- an indented sheet, floating on water;
- a flat circular sheet, strongly attracted to a sphere;
- a naturally-spherical thin sheet, floating on water.

Common features: membrane and bending effects interact with geometry, producing effects not seen in examples discussed earlier.

An indented sheet, floating on water



Flat film floating on water, poked from below
(relaxed problem: Vella et al, PRL 2015;
wrinkling scale: Paulsen et al, PNAS 2016).

Recall from our discussion of tension-driven wrinkling (eg the annulus problem) that

(a) relaxed pbm gave leading-order energy and macroscopic features;

(b) excess energy is from competition btwn bending & membrane terms.

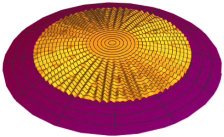
When geometry requires wrinkles to bend, this framework persists but the nature of the competition is different.

Wrinkles don't like to bend (same reason corrugated cardboard is stiff). This effect is was absent in annulus problem, but dominates in the indentation example. Analysis by Paulsen et al is compelling but ansatz-based (we do not yet have an ansatz-free understanding).

A flat circular sheet, strongly attracted to a sphere

The situation changes if the geometric constraint is strong enough.

A flat circular sheet, strongly attracted to a sphere
(Bella & Kohn, Phil Trans Roy Soc 2017)



$$E_h = \int \left| e(u) + \tfrac{1}{2}\nabla w \otimes \nabla w \right|^2 \, dx + h^2 \int |\nabla\nabla w|^2 \, dx + h^{-2} \int \left| w + \frac{|x|^2}{2R} \right|^2 dx$$

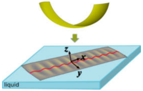
Each circle $r =$ const in sheet avoids compression by wrinkling.

- Optimization of substrate and bending sets length scale; assoc energy is order one.
- Sheet stretches radially, to reduce cost of wrinkling; optimization of radial stretching determines leading-order energy.
- Scale of wrinkling is fixed, so number of wrinkles changes with r. This effect dominates next-order asymptotics in h.

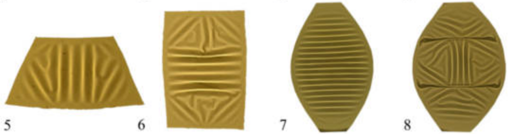
A naturally-spherical thin sheet, floating on water

Breaking news: talks by Eleni Katifori (MS55, Wed 5:00) and Ian Tobasco (MS66, Thurs 9:30).

Sheets cut from spherical shells and placed on water (Albarrán et al, arxiv:1806.03718)



Sheets with the same natural curvature but different shapes show very different wrinkling patterns.

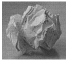


- This is geometrically-driven compressive wrinkling.
- What sets dirn of wrinkling? Is there a variational pbm that determines the leading order energy? (An elegant answer is just now emerging.)

Stepping back

- Variational viewpoint, with thickness as a small parameter: the elastic energy of a sheet is like a Landau theory from physics: a nonconvex membrane energy, regularized by a higher order term (bending) with a small coefficient.
- We're interested in $h \to 0$. Bifurcation diagrams are difficult in this regime. Energy scaling laws provide an alternative.
- Focus on energy scaling law permits rigorous analysis.
- We also get insight about patterns, e.g. whether they achieve the optimal scaling. (But note: patterns in nature can be *local* minima.)
- Analysis is useful, as a complement to simulation. Simulation shows *how* patterns form; ansatz-free lower bounds explain *why* patterns form.

Image Credits



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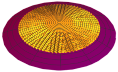

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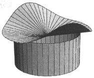

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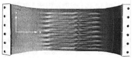

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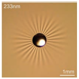

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Credits, cont'd

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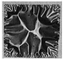

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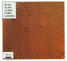

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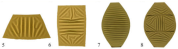

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