A Variational Perspective on Wrinkling due to Geometric Incompatibility

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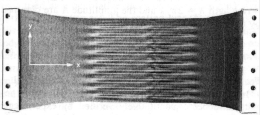
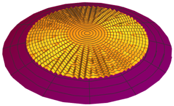
A variational perspective on wrinkling





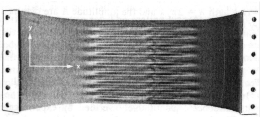
Recall that the calculus of variations has always been driven by challenges from mechanics and geometry

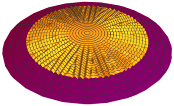
- 1D calculus of variations $\Leftrightarrow$ action minimization, geodesics
- minimal surfaces $\Leftrightarrow$ soap films
- harmonic maps $\Leftrightarrow$ liquid crystals

Wrinkling is similar, but also different:

- similar: patterns come from energy minimization
- similar: there's a rich body of physics literature
- different: there's a small parameter h, and the scale of wrinkling tends to 0 as $h \to 0$. Weak rather than strong convergence.

Why study wrinkling?



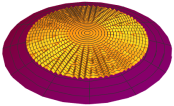


Physical context: rich phenomenology, readily observed; potential applications eg to metrology and device design; provocative analogies to other physical systems where defects and patterns form (eg liquid crystals, ferromagnets, martensitic phase transformation).

Mathematical context: elastic energy is very nonconvex, resembling a Landau theory from condensed matter physics; we see patterns – but how to describe and analyze them? A developing chapter in the calc of varns: “energy-driven pattern formation.”

The big picture

Wrinkling patterns can be complex. What determines their character? How should we even describe them?



Energy minimization as an approach:

- Stable equilibria are local minima of a variational problem (elastic energy plus terms assoc to loads).
- Do the patterns we see achieve (more or less) the global min? (If not – that too would be interesting.)
- To proceed, we need to know: how does min energy depend on the problem's parameters? What is required of a wrinkling pattern, to approach the min energy?
- Minimization within an ansatz is familiar; it gives an upper bound on the minimum energy.
- To complete the story, we need matching lower bounds. They demonstrate the adequacy of the ansatz, and help us understand what drives the patterns.

Talk plan

Not a survey – rather, a story – about wrinkling due to biaxial compression or geometric incompatibility.

(1) Background

- the energy of a thin sheet as a Landau theory

(2) Compression-induced wrinkling

- herringbones and labyrinths (K & Hoai-Minh Nguyen, JNLS 2013)

(3) Geometry-induced wrinkling with tension

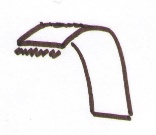
- flat sheet conforming to a round surface (Peter Bella & K, Phil Trans Roy Soc 2017)

(4) Geometry-induced wrinkling with asymptotic isometry

- a curved shell conforming to a flat surface (dramatic progress by Ian Tobasco, arXiv:1906.02153)

Background: thin elastic sheets (fully nonlinear)

Paper is familiar, but already interesting.



Fully nonlinear viewpoint: Configuration of a sheet of paper is a map $g : \Omega \to \mathbb{R}^3$, where $\Omega \subset \mathbb{R}^2$ is the (undeformed) sheet's shape.

Elastic energy consists of membrane energy and bending energy (plus terms assoc to loads or bdry conds).

Membrane energy reflects in-plane stretching or compression of sheet; hence proportional to sheet thickness h. Integrand depends on "principal stretches" (eigenvalues of 2×2 matrix $(Dg^T Dg)^{1/2}$), derivable from 3D elastic law. Key feature: it prefers isometry. Simple example:

$$\text{Membrane energy} = h \int_\Omega \|(Dg^T Dg)^{1/2} - I\|^2 \, dx$$

Backround: thin elastic sheets (fully nonlinear)

Bending energy reflects effect of curvature: if midplane is isometric, parallel surfaces won't be!

Lack of isometry grows linearly with distance to midplane. So

$$\text{Bending energy} = ch^3 \int_\Omega \kappa_1^2 + \kappa_2^2 \, dx$$

where κ_1, κ_2 are principal curvatures of (deformed) sheet.

The energy per unit thickness

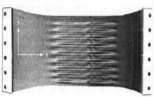
$$E_h = \int_\Omega \|(Dg^T Dg)^{1/2} - I\|^2 \, dx + ch^2 \int_\Omega \kappa_1^2 + \kappa_2^2 \, dx$$

is like a Landau theory from condensed matter physics: a nonconvex term regularized by a higher-order singular perturbation.

After non-dimensionalization: for a sheet with extent L and thickness t, Ω is its normalized shape and h is the eccentricity t/L.

Background: thin elastic sheets (weakly nonlinear)

Some problems involve large rotations (eg the Mobius strip); such problems require a (geometrically) nonlinear treatment. However, for wrinkling a weakly nonlinear (Föppl-von Kármán) model is adequate.



In the simplest setting – wrinkling of a macroscopically flat sheet with Poisson's ratio zero – the weakly nonlinear theory uses

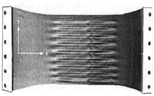
$w =$ out-of-plane displacement and $u =$ in-plane displacement.

The membrane and bending energies are

$$h \int \left| e(u) + \tfrac{1}{2} \nabla w \otimes \nabla w \right|^2 \, dx + h^3 \int |\nabla \nabla w|^2 \, dx$$

where $e_{ij}(u) = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ is the linear strain assoc to u, and $(\nabla w \otimes \nabla w)_{ij} = \partial_i w \, \partial_j w$.

Background: thin elastic sheets (weakly nonlinear)



$$\text{membrane+bending} = h \int_\Omega \|e(u) + \tfrac{1}{2}\nabla w \otimes \nabla w\|^2 \, dx + h^3 \int_\Omega \|\nabla\nabla w\|^2 \, dx$$

The bending term is natural: in small-slope approxn, principal curvatures of graph of w are eigenvalues of $\nabla\nabla w$.

The membrane term is obtained by substituting

$$g(x_1, x_2) = [x_1 + u_1(x_1, x_2), x_2 + u_2(x_1, x_2), w(x_1, x_2)]$$

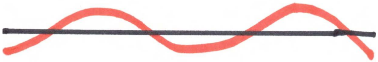
into the fully nonlinear theory, then expanding to leading order (assuming ∇u and ∇w are small).

Explaining the membrane term

To see why the membrane term is

$$h \int \left| e(u) + \tfrac{1}{2} \nabla w \otimes \nabla w \right|^2 \, dx,$$

consider the simpler case of a 1D elastic string in the plane.



Then reference domain is a line segment, elastic displacement is $u_1(x_1)$, and out-of-plane displacement is $w(x_1)$. Assoc nonlinear map

$$g(x_1) = [x_1 + u_1(x_1), w(x_1)]$$

has arclength element

$$\begin{aligned} |g'| &= [(1 + \partial_1 u_1)^2 + (\partial_1 w)^2]^{1/2} \\ &\approx 1 + \partial_1 u_1 + \tfrac{1}{2}(\partial_1 w)^2 \end{aligned}$$

so the square of the strain is

$$(|g'| - 1)^2 \approx \left| \partial_1 u_1 + \tfrac{1}{2}(\partial_1 w)^2 \right|^2$$

Talk plan

(1) Background

(2) Compression-induced wrinkling: herringbones and labyrinths
(work with Hoai-Minh Nguyen, JNLS 2013)

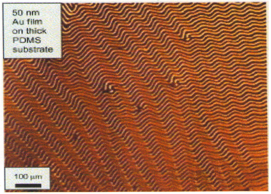
(3) Geometry-induced wrinkling with tension

(4) Geometry-induced wrinkling with asymptotic isometry

Herringbones and labyrinths

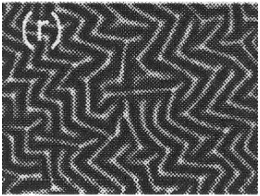
- stretch a polymer layer (biaxially)
- deposit the film
- release the polymer
- film buckles to avoid compression

Commonly seen patterns: herringbones and labyrinths



gold on pdms

Chen & Hutchinson, Scripta Mat 2004



amorphous

Lin & Yang, Appl Phys Lett 2007

Herringbone for crystalline films; labyrinth for amorphous ones.

The herringbone – E_h and the scaling law

Main result: herringbone achieves optimal energy scaling law. (Status of labyrinth is unclear.)

Energy per unit area is

$$E_h = \frac{h}{L^2}\int_{[0,L]^2} |e(u) + \tfrac{1}{2}\nabla w \otimes \nabla w - \eta I|^2 \, dx\, dy + \frac{h^3}{L^2}\int_{[0,L]^2} |\nabla\nabla w|^2 \, dx\, dy + \frac{\alpha_s}{L^2}\left(\|u\|^2_{H^{1/2}} + \|w\|^2_{H^{1/2}}\right)$$

where

$$\eta I = \text{compression of unwrinkled film}$$
$$\alpha_s = \text{ratio of substrate vs film stiffness}$$
$$\text{final term} = \text{elastic energy of substrate}$$

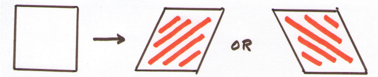
Energy scaling law:

$$\min E_h \sim \min\{\eta^2 h, \alpha_s^{2/3}\eta h\} \sim \min\{\text{flat state, herringbone}\}$$

The herringbone – sketch of the upper bound

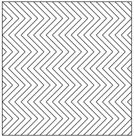
Film wants to expand (isotropically) relative to substrate.

- 1D wrinkling expands only transverse to the wrinkles
- a simple shear expands one diag dirn, compresses the other
- shear combined with wrinkling achieves isotropic expansion



Substrate prohibits large deformation; therefore film mixes the shear-combined-with-wrinkling variants. (Bdry layers introduce some membrane energy, not significant.) Pattern has two length scales:

- The smaller one (the scale of the wrinkling) is set by competition between the *bending* term and the *substrate energy of w*.
- The larger one (scale of the phase mixture) must be s.t. *substrate energy of u is insignificant*. (It is not fully determined.)



The herringbone – sketch of the lower bound

Claim: For any L-periodic u and w, $E_h \geq C \min\{\eta^2 h, \alpha_s^{2/3} \eta h\}$.

Let's take $L = 1$ for simplicity. We'll use only that

$$\text{membrane term} \geq h \int \left| \partial_1 u_1 + \tfrac{1}{2}(\partial_1 w)^2 - \eta \right|^2 \, dx$$

$$\text{bending term} = h^3 \|\nabla\nabla w\|_{L^2}^2, \quad \text{and} \quad \text{substrate term} \geq \alpha_s \|w\|_{H^{1/2}}^2.$$

CASE 1: If $\int |\nabla w|^2 \ll \eta$ then membrane energy is $\gtrsim \eta^2 h$, since $\partial_1 u_1$ has mean 0.

CASE 2: If $\int |\nabla w|^2 \gtrsim \eta$ use the interpolation inequality

$$\|\nabla w\|_{L^2} \lesssim \|\nabla\nabla w\|_{L^2}^{1/3} \|w\|_{H^{1/2}}^{2/3}$$

to see that

$$\begin{aligned} \text{Bending + substrate terms} &= h^3 \|\nabla\nabla w\|^2 + \tfrac{1}{2}\alpha_s \|w\|_{H^{1/2}}^2 + \tfrac{1}{2}\alpha_s \|w\|_{H^{1/2}}^2 \\ &\gtrsim \left(h^3 \|\nabla\nabla w\|^2 \alpha_s^2 \|w\|_{H^{1/2}}^4 \right)^{1/3} \\ &\gtrsim h \alpha_s^{2/3} \|\nabla w\|_{L^2}^2 \gtrsim h \alpha_s^{2/3} \eta \end{aligned}$$

using arith mean/geom mean inequality.

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using arith mean/geom mean inequality.

Talk plan

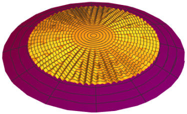
(1) Background and context

(2) Compression-induced wrinkling

(3) Geometry-induced wrinkling with tension:
a flat sheet conforming to a round surface
(work with Peter Bella, Phil Trans Roy Soc 2017, building on Hohlfeld & Davidovitch, PRE 2015)

(4) Geometry-induced wrinkling with asymptotic isometry

A flat sheet conforming to a round surface



A flat circular sheet, wrapped around a sphere; energy per unit thickness is

$$E_h = \int \left|e(u) + \tfrac{1}{2}\nabla w \otimes \nabla w\right|^2 dx + h^2 \int |\nabla\nabla w|^2 \, dx + \alpha_s h^{-2} \int \left| w + \frac{|x|^2}{2R} \right|^2 dx$$

Modeling choices:

- Substrate is strong: $\alpha_s h^{-2} \to \infty$ as $h \to 0$.
- Slip is permitted: substrate term requires $w \approx -\frac{1}{2}|x|^2/R$ (small slope approx to sphere), but doesn't constrain u.

Expected behavior: since sheet is not isometric to sphere, each circle $r =$ const wrinkles to avoid compression.

Main result: we found the leading-order energy $\mathcal{E}_0$ and estimated the next-order correction:

$$\mathcal{E}_0 + Ch \leq \min E_h \leq \mathcal{E}_0 + C' h|\log h|$$

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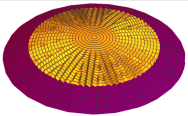

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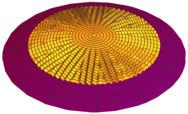
The leading-order energy

$$E_h = \int \left|e(u) + \tfrac{1}{2}\nabla w \otimes \nabla w\right|^2 \, dx + h^2 \int |\nabla\nabla w|^2 \, dx + \alpha_s h^{-2} \int \left| w + \frac{|x|^2}{2R} \right|^2 \, dx$$

(1) Expect circle $|x| = r$ to map to a circle on the sphere. It must wrinkle, since image circle has smaller arclength.

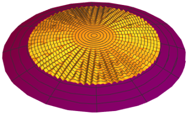


(2) Optimization of substrate and bending sets length scale; assoc energy is of order one (assuming α_s is fixed as $h \to 0$). Magnitude depends on amount of arclength to be “wasted by wrinkling”.

(3) Sheet stretches radially, to reduce cost of wrinkling. Tradeoff btwn membrane term (cost of radial stretching) and substrate+bending (cost of wrinkling) sets leading-order energy. Optimization of tradeoff leads to 1D variational problem for the radial stretch u_r.

Flat sheet on a sphere – the order-h correction

What controls the correction $\min E_h - \mathcal{E}_0$?



- Optimal scale of wrinkling depends only on h and α_s (not on r).
- This means that wrinkles must branch! (As r increases, the number of wrinkles at radius r must grow linearly with r.) Cost of branching is at heart of correction.
- Rough idea: scale of wrinkling need not be exactly optimal; since we expect excess energy of order h, it has to change only when cost of suboptimality is of order h.
- After analysis (and simplifying a bit): number of wrinkles at radius r is a piecewise-const approx to a linear function, taking values that are integer multiples of $h^{-1/2}$.

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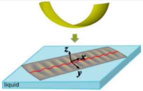
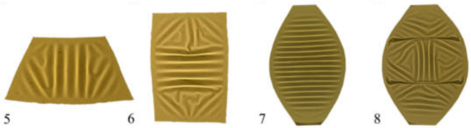
(4) Geometry-induced wrinkling with asymptotic isometry:
a curved shell conforming to a flat surface
(Ian Tobasco, arXiv:1906.02153)

A curved shell conforming to a flat surface

A piece of a spherical shell is placed on water (Aharoni et al, Nature Commun 2017; Albarrán et al, arXiv:1806.03718)



Different shapes develop very different wrinkling patterns. Unlike previous example, tension seems to play no role.



Expts and some ansatz-based theory: groups of E. Katifori (Penn), J. Paulsen (Syracuse), B. Davidovitch (U Mass). Recently, striking progress by Ian Tobasco:

(1) Identification of min energy, in a suitable parameter regime.

(2) An asymptotic var'l pbm for the macroscopic displacement u.

(3) The asymptotic pbm is convex, and its dual is explicitly solvable in many cases. Optimality conds $\Rightarrow$ predictions about wrinkling.

A curved shell conforming to a flat surface

Tobasco's starting point:

$$E_{b,k} = \int_\Omega |e(u) + \tfrac{1}{2}\nabla w \otimes \nabla w - \tfrac{1}{2}\nabla p \otimes \nabla p|^2 + b\int_\Omega |\nabla\nabla w|^2 + k\int_\Omega w^2;$$

here the graph of p is the natural shape of the shell, Ω is geometry of cutout, $b = h^2$, and k reflects strength of substrate term (which for water is gravitational energy).

We expect asymptotic isometry, so the substrate term should not be too strong:

$$b \to 0, \quad k \to \infty, \quad \text{with} \quad bk \to 0$$

(different from our sheet-on-sphere example, where $bk = \alpha_s$ was held constant, leading to order-one tension.) Actually, one needs a little more: $b^{-2/3} \ll k \ll b^{-1}$.

Theorem: In this regime, $\frac{E_{b,k}}{4\sqrt{bk}}$ Γ-converges to

$$\int_\Omega \tfrac{1}{2}|\nabla p|^2\,dx - \int_{\partial\Omega} u \cdot \nu\,d\sigma$$

subject to the constraint $e(u) \leq \frac{1}{2}\nabla p \otimes \nabla p$.

Convex duality and wrinkling

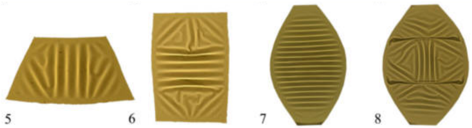
The limit problem

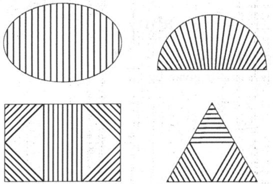
$$\min_{e(u)\leq\frac{1}{2}\nabla p\otimes\nabla p} \int_\Omega \tfrac{1}{2}|\nabla p|^2\,dx - \int_{\partial\Omega} u\cdot\nu\,d\sigma$$

has an intuitive interpretation: membrane term is strongest, so extension is prohibited; when wrinkling occurs, interaction of bending and substrate terms produces an effective surface tension.

Limit is convex, so it has a convex dual. When $\det\nabla\nabla p > 0$ (shell has pos curvature) or $\det\nabla\nabla p < 0$ (shell has neg curvature), the dual has a unique solution σ, which depends only on Ω and can be made explicit in many cases.

Solution of dual is, roughly speaking, Lagrange multiplier for the constraint. So: where σ has rank one, it tells us dirn of wrinkling.





The herringbone revisited

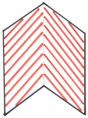
To understand idea of Tobasco's asymptotic variational problem, consider the herringbone in his regime:

$$E_{b,k} = \int_{[0,1]^2} |e(u) + \tfrac{1}{2}\nabla w \otimes \nabla w - \eta I|^2 + b \int_{[0,1]^2} |\nabla\nabla w|^2 + k \int_{[0,1]^2} w^2;$$

Claim: In Tobasco's limit (with η held fixed, and with periodic bc),

$$\frac{\min E_{b,k}}{4\sqrt{bk}} \to 2\eta.$$

Proof – Part 1: membrane term $\to$ 0 in this limit.
In fact, a herringbone construction (with just two stripes) shows $E_{b,k}/4\sqrt{bk}$ is bounded above.



Since $\sqrt{bk} \to 0$, this means membrane term must vanish in the limit,

$$e(u) + \tfrac{1}{2}\nabla w \otimes \nabla w - \eta I \to 0 \quad \text{in } L^2,$$

so (taking the trace)

$$\operatorname{div} u + \tfrac{1}{2}|\nabla w|^2 - 2\eta \to 0.$$

The herringbone revisited

Proof – Step 2: bending and substrate terms of $E/4\sqrt{bk}$ tend to 2η.

In fact, bending and substrate terms of $E/4\sqrt{bk}$ are

$$\frac{1}{4}\sqrt{b/k}\int_{[0,1]^2}|\nabla\nabla w|^2+\frac{1}{4}\sqrt{k/b}\int_{[0,1]^2}w^2.$$

Since w is periodic, $\int|\nabla\nabla w|^2=\int(\Delta w)^2$; so integrating the identity

$$\sqrt{b/k}\,(\Delta w)^2+\sqrt{k/b}\,(w)^2=2|\nabla w|^2+\left|(b/k)^{1/4}\Delta w+(k/b)^{1/4}w\right|^2-2\,\mathrm{div}\,(w\nabla w)$$

gives

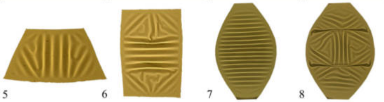
$$\begin{aligned}\frac{1}{4}\sqrt{b/k}\int|\nabla\nabla w|^2+\frac{1}{4}\sqrt{k/b}\int w^2\\ &=\frac{1}{2}\int|\nabla w|^2+\frac{1}{4}\int\left|(b/k)^{1/4}\Delta w+(k/b)^{1/4}w\right|^2\\ &\geq\frac{1}{2}\int|\nabla w|^2,\end{aligned}$$

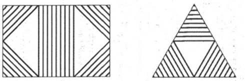
with asymptotic equality for wrinkling at proper scale.

Finally, using Step 1 ($\mathrm{div}\,u+\frac{1}{2}|\nabla w|^2-2\eta\to 0$) and periodicity of u, lower bound becomes

$$\int 2\eta-\mathrm{div}\,u=2\eta.$$

Floating shells vs the herringbone problem

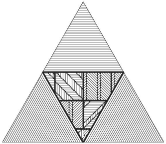




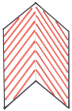
- For Tobasco, pbm has natural bc (not periodic), and the pre-strain is $\frac{1}{2}\nabla p \otimes \nabla p$ (not ηI). So his lower bound is

$$\int_\Omega \tfrac{1}{2}|\nabla p|^2 - \operatorname{div} u = \int_\Omega \tfrac{1}{2}|\nabla p|^2 - \int_{\partial\Omega} u \cdot \nu.$$

- Local compression can be uniaxial or biaxial. In biaxial region, upper-bound ansatz uses piecewise-constant approxn then herringbone-like construction.

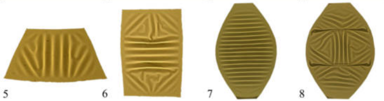


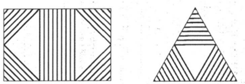
- Extra condition $b^{-2/3} \ll k \ll b^{-1}$ is needed so "boundary layers" of the herringbones introduce negligible additional energy.



- Regions with uniaxial compression (1D wrinkling) are experimentally robust. Those with biaxial compression are not.

How does Tobasco predict 1D wrinkling domains?





Tobasco predicts regions of 1D wrinkling. How?

Limiting variational problem is convex:

$$\min_{e(u) \leq \frac{1}{2}\nabla p \otimes \nabla p} \int_{\Omega} \frac{1}{2}|\nabla p|^2 - \operatorname{div} u$$

Its dual involves a divergence-free vector field σ, which can be viewed as a Lagrange multiplier for the constraint. Surprisingly, the dual can be solved explicitly in many cases.

- Where σ has rank one, only 1D wrinkling is possible (normal to null direction of σ).
- Where $\sigma = 0$ we get no prediction (expect 2D wrinkling and/or nonuniqueness).

Note: If $\det \nabla\nabla p \neq 0$ then σ cannot have rank two on an open set, since elastic compatibility rules out $e(u) = \frac{1}{2}\nabla p \otimes \nabla p$.

Formal duality

$$\inf_{e(u)\le\frac{1}{2}\nabla p\otimes\nabla p} \int_\Omega \tfrac{1}{2}|\nabla p|^2 - \operatorname{div} u$$

$$= \inf_{e(u)} \sup_{\sigma\ge 0} \int_\Omega \langle \sigma, e(u) - \tfrac{1}{2}\nabla p\otimes\nabla p\rangle + \tfrac{1}{2}|\nabla p|^2 - \operatorname{div} u$$

$$= \sup_{\sigma\ge 0} \inf_{e(u)} \int_\Omega \langle \sigma - I, e(u)\rangle - \tfrac{1}{2}\langle\sigma, \nabla p\otimes\nabla p\rangle + \tfrac{1}{2}|\nabla p|^2$$

$$= \sup_{\substack{\sigma\ge 0\\ \mathrm{div}\,\sigma=0 \text{ in } \Omega\\ \sigma\cdot\nu=\nu \text{ at } \partial\Omega}} \int_\Omega \langle I-\sigma, \tfrac{1}{2}\nabla p\otimes\nabla p\rangle$$

Some issues: σ can be discontinuous; $\sigma\cdot\nu$ is only defined weakly at $\partial\Omega$; explicit solvability is not yet clear.

Key to resolution: represent σ by an Airy stress function,

$$\sigma = \begin{pmatrix} \partial_{22}\phi & -\partial_{12}\phi \\ -\partial_{12}\phi & \partial_{11}\phi \end{pmatrix}$$

Duality, cont'd

When $\sigma = \begin{pmatrix} \partial_{22}\phi & -\partial_{12}\phi \\ -\partial_{12}\phi & \partial_{11}\phi \end{pmatrix}$ we have:

(a) $\text{div}\,\sigma = 0$ and $\sigma \geq 0$ iff ϕ is continuous and convex.

(b) Bdry condition $\sigma \cdot \nu = \nu$ can be imposed by taking $\phi = \frac{1}{2}|x|^2$ outside Ω (and asking that the extension be cont's & convex).

(c) Integration by parts: if $\psi = \frac{1}{2}|x|^2 - \phi$ is Airy stress fn for $I - \sigma$ then for any symmetric-matrix-valued $\xi(x)$,

$$\int_\Omega \left\langle \begin{pmatrix} \partial_{22}\psi & -\partial_{12}\psi \\ -\partial_{12}\psi & \partial_{11}\psi \end{pmatrix}, \xi \right\rangle = \int_\Omega \psi(\partial_{22}\xi_{11} + \partial_{11}\xi_{22} - 2\partial_{12}\xi_{12})$$

Applying this to $\xi = \frac{1}{2}\nabla p \otimes \nabla p$ puts the dual problem in the convenient form:

$$\sup_{\substack{\sigma \geq 0 \\ \text{div}\,\sigma = 0 \text{ in } \Omega \\ \sigma\cdot\nu = \nu \text{ at } \partial\Omega}} \int_\Omega \langle I - \sigma, \tfrac{1}{2}\nabla p \otimes \nabla p \rangle = \sup_{\substack{\phi \text{ conts and convex} \\ \phi = \frac{1}{2}|x|^2 \text{ outside } \Omega}} \int_\Omega (\phi - \tfrac{1}{2}|x|^2) \det \nabla\nabla p.$$

Explicit solutions

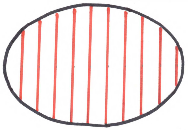
When the Gaussian curvature of the shell is pointwise positive ($\det \nabla\nabla p > 0$), the optimal ϕ for the dual problem

$$\sup_{\substack{\phi \text{ conts and convex} \\ \phi = \frac{1}{2}|x|^2 \text{ outside } \Omega}} \int_\Omega (\phi - \tfrac{1}{2}|x|^2) \det \nabla\nabla p$$

is the largest convex function that equals $\frac{1}{2}|x|^2$ outside Ω. Examples:

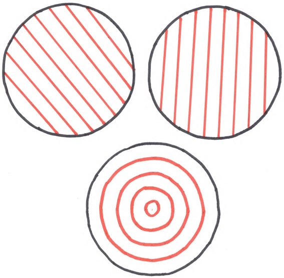
Warmup: in 1D, consider $\frac{1}{2}x^2$ off $\Omega = (-c, c)$: the largest convex extension is constant on the interval.

An ellipse: In 2D, let $\Omega = \{x_1^2/a^2 + x_2^2/b^2 < 1\}$ with $b < a$. Apply warmup to each slice $x_1 = \text{const}$ to see that best ϕ is a function of x_1 only: $\phi = \frac{1}{2}[b^2 + (1 - \frac{b^2}{a^2})x_1^2]$. Evidently $\sigma_{22} \neq 0$, so wrinkling is in the x_1 direction.

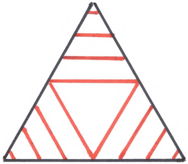


Explicit solutions, cont'd

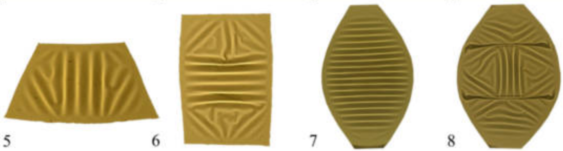
A circle is a degenerate ellipse. Evidently, when Ω is a circle the wrinkling pattern is not unique. In fact $\sigma = 0$ in this case (ϕ is constant). Besides the 1D wrinkling patterns obtained by taking limits of ellipses, there are other patterns as well, including axially-symmetric 1D wrinkling.

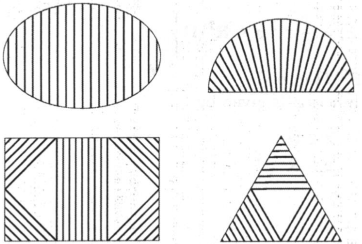


For a triangle, choose coords so $x = 0$ is the center of the inscribed circle. Then warmup calculation (on slices) shows that ϕ is a function of one variable in a triangle near each vertex. In the inner triangle ϕ is constant, $\sigma = 0$, and the theory makes no prediction. (Experimentally, the patterns there are not robust).



Another look at the pictures





Stepping back

- Variational viewpoint, with thickness as a small parameter: the elastic energy of a sheet is like a Landau theory – a nonconvex membrane energy, regularized by a higher order term (bending) with a small coefficient.
- Problems with biaxial compression or geometric incompatibility are especially interesting, because even the direction of wrinkling is unclear.
- We're interested wrinkling patterns. Bifurcation diagrams are difficult in this regime. Energy minimization provides an alternative (though nature may find *local* minima.)
- Analysis is useful, as a complement to simulation. Simulation shows *how* patterns form; ansatz-free lower bounds explain *why* they form.
- Tobasco's work especially striking, because it explains wrinkling patterns that were previously mysterious.

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Credits

Funding from NSF is gratefully acknowledged.

Image credits:

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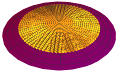

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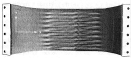

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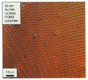

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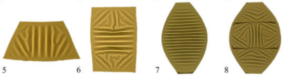

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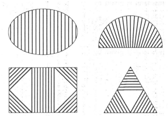

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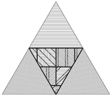

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