Wrinkling of thin elastic sheets – Lecture 2: Mathematical context and the relaxed energy

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Goals for Lecture 2

- Mathematical perspective on nonconvex variational problems with regularizing singular perturbations
  - A transparent 1D warmup
  - Some 2D and 3D examples
- Wrinkling as microstructure: the relaxed membrane energy
  - The meaning of “relaxation”
  - Implementation for our membrane energies

The big picture

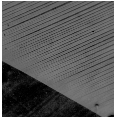
The elastic energy of a thin sheet has form

$$E_h = \begin{bmatrix} \text{nonconvex membrane energy} \\ \text{(prefers isometry)} \end{bmatrix} + h^2 \begin{bmatrix} \text{bending energy} \\ \text{(penalizes change in slope)} \end{bmatrix}$$

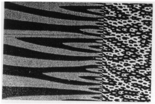
We have argued (just handwaving so far) that the patterns we see are often due to competition between these two terms ("frustration").

This idea is not unique to thin sheets. Many systems are described by "Landau theories" with similar structure, eg liquid crystals, magnetic materials, shape-memory materials. Past studies in the spirit of these lectures include:

Branching of twins near an interface between austenite and twinned martensite Figure: R. James. Theory: Kohn & Müller, CPAM, 1994.

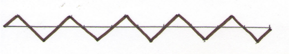


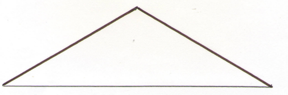
Branching of magnetic domains in a uniaxial ferromagnet. Figure: Hubert and Schäfer. Theory: Choksi, Kohn, Otto, CMP, 1999.



A transparent 1D warmup

$$\min_{u(0)=u(1)=0} \int_0^1 (u_x^2 - 1)^2 + \varepsilon^2 u_{xx}^2 + \alpha u^2 \, dx$$





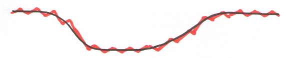
- When $\varepsilon = 0, \alpha > 0$, min value is 0, not attained. Min sequence has "$u_x = \pm 1$ with prob $1/2$ each."
- When $\varepsilon > 0$, min scales like $\varepsilon^{2/3}\alpha^{1/3}$, since for sawtooth with N teeth, value is about $\sim \varepsilon N + \alpha N^{-2}$. Best $N \sim (\alpha/\varepsilon)^{1/3}$.
- Case $\varepsilon > 0, \alpha = 0$ is different: just one tooth; min value is $c_0 \varepsilon$.
- Optimal profile of "tooth" (which determines c_0) found by minimizing $\int \varepsilon^{-1}(w^2 - 1)^2 + \varepsilon w_x^2$ subject to $w \to \pm 1$ as $x \to \pm\infty$. Exactly solvable.

Singular perturbation induces defects and organizes microstructure.

Relaxation in the 1D setting

Now consider the less homogeneous problem

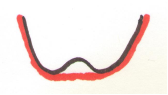
$$\min \int_0^1 (u_x^2 - 1)^2 + |u - g(x)|^2 + \varepsilon^2 u_{xx}^2 \, dx$$



An approx soln can be obtained in two steps:

STEP 1: Solve the relaxed problem:

$$\min \int_0^2 f(u_x) + |u - g|^2 \, dx$$



where f is "convexification" of $(u_x^2 - 1)^2$

This gives the $\varepsilon \to 0$ limit of min energy. Being (strictly) convex, the relaxed problem has a (unique) solution.

STEP 2: Put back the oscillations, by making $u_x = \pm 1$ in regions where soln of relaxed problem has $|\text{slope}| < 1$.

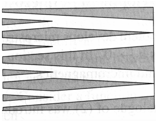
2D is richer than 1D

A favorite example:

$$\min_{u=0 \text{ at } x=0} \int (u_y^2 - 1)^2 + u_x^2 + \varepsilon^2 |\nabla\nabla u|^2$$

or

$$\min_{\substack{u=0 \text{ at } x=0 \\ u_y=\pm 1}} \int u_x^2 + c_0 \varepsilon |u_{yy}|$$



- Simple model for branching of twins near austenite interface
- Like 1D example (in y), but microstructure is required by bdry cond rather than a lower-order term
- Relaxed problem is (essentially) min $\int u_x^2$ subject to $u = 0$ at $x = 0$. Unique soln is $u = 0$.
- Microstructural length scale depends on x (finer near $x = 0$)
- Much is known (Kohn-Müller, Conti), esp for the sharp-interface version
- See TA session 2, problem 3.

2D is richer than 1D, cont'd

Two more favorite examples:

The Aviles-Giga functional

$$\min_{u=0 \text{ at } \partial\Omega} \int_\Omega \varepsilon^{-1}(|\nabla u|^2 - 1)^2 + \varepsilon|\nabla\nabla u|^2 \, dx$$

where $\Omega \subset \mathbb{R}^2$ and $u : \Omega \to \mathbb{R}$. Nonconvex term prefers $|\nabla u| = 1$. Minimizers have walls (graph of u has folds). No microstructure. Connections to magnetic thin films and smectic liquid crystals, see eg my article in Proc ICM 2006.

Ginzburg-Landau vortices

$$\min_{\text{bc}} \int_\Omega \varepsilon^{-1}(|v|^2 - 1)^2 + \varepsilon|\nabla v|^2 \, dx$$

where $\Omega \subset \mathbb{R}^2$ and $v : \Omega \to \mathbb{R}^2$. More freedom than Aviles-Giga, since v need not be curl-free. Minimizers have point defects at which $\deg(v) = \pm 1$. Connections to type-II superconductors, see eg books by Bethuel, Brezis, & Helein and Sandier & Serfaty.

Turning now to thin elastic sheets

Recall our warmup problem:

$$\min \int_0^1 (u_x^2 - 1)^2 + |u - g(x)|^2 + \varepsilon^2 u_{xx}^2 \, dx$$

and our 2-step approach to understanding its limiting behavior as $\varepsilon \to 0$.

What is the analogue for a thin elastic sheet, described by the var'l problem

$$\min \text{ (membrane energy)} + h^2 \text{(bending energy)} + \text{(loads)?}$$

Conceptual answer:

Step 1: Identify the relaxed membrane energy, ie the min energy achievable using (if necessary) "infinitesimal wrinkling". (Ignore bending).

Step 2: Keeping the loads and bc the same, replace the membrane energy by its relaxation, set $h = 0$, and solve.

Step 3: To account for $h > 0$, start with the result of Step 2 and "put back the wrinkles."

Easy to visualize, eg for

- hanging drapes, pulled by gravity
- sheet pulled at two opposite sides
- sheet pulled down over a sphere
- mylar balloon, wrinkled at seams

Relaxation, continued

Step 1: Identify the relaxed membrane energy, ie the min energy achievable using (if necessary) "infinitesimal wrinkling". (Ignore bending).

Step 2: Keeping the loads and bc the same, replace the membrane energy by its relaxation, set $h = 0$, and solve.

Step 3: To account for $h > 0$, start with the result of Step 2 and "put back the wrinkles."

- Steps 1 & 2 are useful by themselves: if solution is unique, it tells us the extent and direction of wrinkling (though not the scale).
- Not a new idea: steps 1 & 2 are equivalent to "tension field theory."
- Step 3 is not so simple – it is more or less the focus of my lectures.
- Program is mainly useful for tension-driven wrinkling, since relaxed membrane energy vanishes for biaxial compression.

A characterization of the relaxed energy

For $u : \mathbb{R}^m \to \mathbb{R}^n$, the relaxation of $W(Du)$ is its quasiconvexification,

$$
\begin{aligned}
QW(\xi) &= \min_{u=\xi\cdot x \text{ at} \partial D} \frac{1}{|D|}\int_D W(Dg) \\
&= \text{min avg energy consistent with avg gradient } \xi
\end{aligned}
$$

- There's a well-developed theory (see eg Dacorogna, Direct Methods in the Calculus of Variations; or sec 2 of Kohn & Vogelius, CPAM 40 (1987) 745-777).
- Well-defined: value doesn't depend on choice of domain D.
- In 1D example, relaxation was the convexification (and was therefore explicit). For our membrane energy, the relaxation is again the convexification. But QW is not always convex, and not always easy to find explicitly.

The relaxed membrane energy – nonlinear version

Consider our simple model membrane energy

$$W = |Dg^T Dg - I|^2 = (\lambda_1^2 - 1)^2 + (\lambda_2^2 - 1)^2.$$

where $g : \mathbb{R}^2 \to \mathbb{R}^3$ and λ_i are its principal stretches.

Claim: its relaxation QW is

$$f = (\lambda_1^2 - 1)_+^2 + (\lambda_2^2 - 1)_+^2$$

with the notation $t_+ = \max\{t, 0\}$.

The function f is actually the convexification of W, though this is not so easy to see. I'll prove the claim without proving convexity, by working directly with the definition of the relaxation. Need to show two inequalities:

(1) $QW \leq f$: requires identifying oscillations whose average membrane energy is at most f.

(2) $QW \geq f$: requires showing that regardless of oscillation, average membrane energy is at least f.

$QW \leq f$

Recall: $W = (\lambda_1^2 - 1)^2 + (\lambda_2^2 - 1)^2$ and $f = (\lambda_1^2 - 1)_+^2 + (\lambda_2^2 - 1)_+^2$.

We must identify oscillations st average membrane energy is f.

Case 1: $\lambda_1 > 1$ and $\lambda_2 > 1$: nothing to do (no oscillation needed).

Case 2: $\lambda_1 > 1$ and $\lambda_2 \leq 1$: use a 1D pattern of folds, with a transition layer near the boundary. (Note: folds need not be sharp; they could eg be sinusoidal).

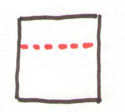
Case 3: $\lambda_1 < 1$ and $\lambda_2 < 1$: repeat this calculation with two distinct length scales. (Other constructions are also possible, eg using the Miura origami pattern.)

$QW \geq f$

Recall: $W = (\lambda_1^2 - 1)^2 + (\lambda_2^2 - 1)^2$ and $f = (\lambda_1^2 - 1)_+^2 + (\lambda_2^2 - 1)_+^2$.

Must show that regardless of oscillation, avg membrane energy is at least f. Sufficient to consider domain $D = [0,1]^2$ and maps $g : D \to \mathbb{R}^3$ st $g = (\lambda_1 x_1, \lambda_2 x_2, 0)$ at ∂D.

Idea: when $\lambda_1 > 1$, horizontal lines must be stretched; when $\lambda_2 > 1$, vertical lines must be stretched.



Implementation uses just Jensen's inequality and fundamental theorem of calculus.

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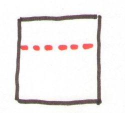


Since $(|\partial_1 g_1|^2 - 1)_+^2$ is a convex function of $\partial_1 g_1$, Jensen gives

$$(\lambda_1^2 - 1)_+^2 \leq \int_0^1 (|\partial_1 g_1|^2 - 1)_+^2 \, dx_1 \quad \text{for each } x_2$$

and integration in x_2 gives

$$(\lambda_1^2 - 1)_+^2 \leq \int_D (|\partial_1 g_1|^2 - 1)_+^2$$

But since $|\partial_1 g_1|^2 - 1 = (Dg^T Dg - I)_{11}$ we have

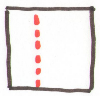
$$(|\partial_1 g_1|^2 - 1)_+^2 \leq (Dg^T Dg - I)_{11}^2$$

so

$$(\lambda_1^2 - 1)_+^2 \leq \int_D (Dg^T Dg - I)_{11}^2.$$

$QW \geq f$ cont'd

Idea: when $\lambda_1 > 1$, horizontal lines must be stretched; when $\lambda_2 > 1$, vertical lines must be stretched.



Similarly (integrating along vertical lines)

$$(\lambda_2^2 - 1)_+^2 \leq \int_D (Dg^T Dg - I)_{22}^2.$$

Adding, we get

$$f \leq \int_D |Dg^T Dg - I|^2 = \int_D W(Dg).$$

Minimizing RHS over all g with the given bc, we get $f \leq QW$.

The relaxed membrane energy – von Karman version

In the von Karman setting the situation is similar, though the algebraic details are different. The membrane energy is

$$W = |e(w) + \tfrac{1}{2}\nabla u_3 \otimes \nabla u_3|^2 = \mu_1^2 + \mu_2^2$$

where μ_1, μ_2 are the eigenvalues of $e(w) + \frac{1}{2}\nabla u_3 \otimes \nabla u_3$.

Claim: Its relaxation QW is

$$f = (\mu_1)_+^2 + (\mu_2)_+^2.$$

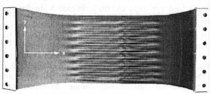
The proof is parallel to the nonlinear case. (But: the convexity of f is somewhat easier in this setting – see TA Session 2 problem 2. This gives alternative proof that $QW \geq f$, using Jensen).

Stepping back

Connection to Lecture 1: for tension-driven wrinkling, relaxed problem tells us where the wrinkling occurs, and even the "arclength that must be wasted by wrinkling" to avoid compression. Results about the length scale of wrinkling when $h > 0$ are in some sense perturbative around this.

- hanging drapes (Vandeparre et al, PRL 2011)

- stretched sheets (Cerda & Mahadevan, PRL 2003)

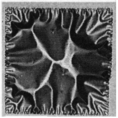


- water drop on floating sheet (Huang et al, Science 2007)

Stepping back – cont'd

For compressive problems, however, the relaxed problem is typically of less use, due to its extreme degeneracy.

- Compression due to thermal mismatch: a thin film bonded to a too-short bdry (Lai et al, J Power Sources 2010)



Some comments

- These methods are not limited to these examples. I have focused on my favorite model membrane energies, but a similar discussion applies for any (isotropic) membrane energy (Pipkin, IMA J Appl Math 36, 1986, 85-99)
- In general, relaxation is different from convexification. The fact that our relaxed problems have all been convex is a bit misleading. In general, for problems of type $W(Du)$ with $u : \mathbb{R}^n \to \mathbb{R}^n$ with $m, n \geq 2$, the relaxed energy need not be convex. A simple example is $W(Du) = \det Du$, which is its own relaxation (though it is not convex).
- Martensitic microstructure can be viewed in these terms. In martensitic transformation, it is natural to ask which macroscopic deformations are achieved by (approximately) stress-free mixtures of the different phases. This is equivalent to asking: suppose $W(Du) \geq 0$ and $W = 0$ at selected sets of the form $SO(3)U_i$, $1 \leq i \leq N$; where is QW equal to 0? In such problems the relaxed energy QW is never convex.

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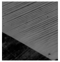
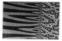
Image credits



R.D. James, Univ of MN



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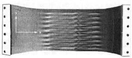

E. Cerda and L. Mahadevan, *Phys Rev Lett* 90 (2003) 074302

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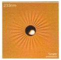

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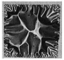

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