Wrinkling of thin elastic sheets – Lecture 4: The herringbone pattern

Robert V. Kohn
Courant Institute, NYU

PCMI, July 2014

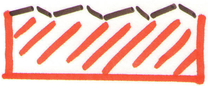
Orientation

- Recall from Lecture 1: *tension-induced* wrinkling and *compression-induced* wrinkling are very different.
- We understand from Lecture 2 that for tension-induced wrinkling the relaxed problem is nontrivial; it determines the wrinkled region and the direction of the wrinkles. Lecture 3 explored an example.
- For compression-induced wrinkling, the relaxed problem is trivial and it provides no guidance. Today's lecture presents work with Hoai-Minh Nguyen on an example of this type.
- For more detail see: R.V. Kohn & H.-M. Nguyen, J Nonlin Sci 23 (2013) 343-362.

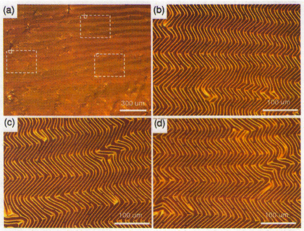
Phenomenology

Wrinkling of thin films compressed by thick, compliant substrates:

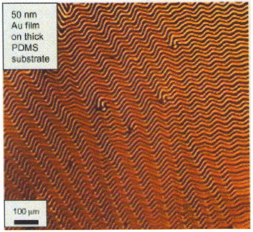
- deposit film at high temp then cool; or
- deposit on stretched substrate then release;
- film buckles to avoid compression



Commonly seen pattern: herringbone



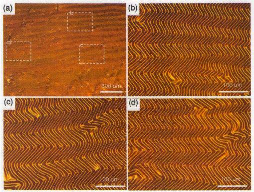
silicon on pdms



gold on pdms

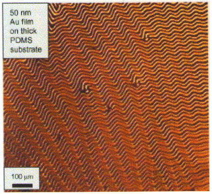
Phenomenology - cont'd

Herringbone pattern when film has some anisotropy, or for specific release histories. Otherwise a less ordered "labyrinth" pattern.



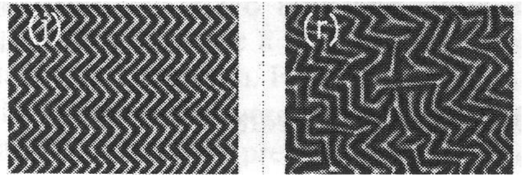
silicon on pdms

Song et al, *J Appl Phys* 103 (2008) 014303



gold on pdms

Chen & Hutchinson, *Scripta Mat* 50 (2004) 797–801

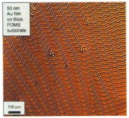


different release histories

Lin & Yang, *Appl Phys Lett* 90 (2007) 241903

The elastic energy

Using the von Karman framework, the energy has three terms:



(1) Membrane energy captures fact that film's natural length is larger than that of the substrate:

$$\alpha_m h \int \left| e(w) + \tfrac{1}{2}\nabla u_3 \otimes \nabla u_3 - \eta I \right|^2 dx\, dy$$

(2) Bending energy captures resistance to bending:

$$h^3 \int |\nabla\nabla u_3|^2\, dx\, dy$$

(3) Substrate energy captures fact that substrate acts as a "spring", tending to keep film flat:

$$\alpha_s \left(\|w\|^2_{H^{1/2}} + \|u_3\|^2_{H^{1/2}} \right)$$

where $\|g\|^2_{H^{1/2}} = \sum |k||\hat{g}(k)|^2$

Membrane energy is proportional to h, and bending to h^3, since substrate energy is proportional to area.

The membrane energy

$$E_{\text{membrane}} = \alpha_m h \int |e(w) + \tfrac{1}{2}\nabla u_3 \otimes \nabla u_3 - \eta I|^2 \, dx\, dy$$

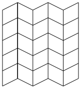
where (w_1, w_2, u_3) is the elastic displacement, and $\eta > 0$ is the misfit (nondimensional but small). Keeping α_m as a parameter permits us to see when the membrane term is important.

- For membrane term to be small, expect $|e(w)| \sim \eta$ and $|\nabla u_3| \sim \sqrt{\eta}$.
- For 1D analogue $\int |\partial_x w_1 + \frac{1}{2}(\partial_x u_3)^2 - \eta|^2 \, dx$, integrand vanishes eg if wrinkling profile is sinusoidal,

$$w_1 = \eta \frac{\lambda}{4} \sin(4x/\lambda), \;\; u_3 = \sqrt{\eta}\lambda \cos(2x/\lambda)$$

- Our problem is 2D, with isotropic misfit ηI;

  membrane term would vanish for piecewise-linear “Miura ori” pattern.



- The herringbone pattern uses sinusoidal wrinkling in two distinct orientations. It does better than the Miura-ori pattern.

The substrate energy

$$E_{\text{substrate}} = \alpha_s \left(\|w\|^2_{H^{1/2}} + \|u_3\|^2_{H^{1/2}} \right)$$

where (w_1, w_2, u_3) are assumed periodic (on some large scale L),

$$\|g\|^2_{H^{1/2}} = \sum |k||\hat{g}(k)|^2$$

and

$$\alpha_s = \text{substrate stiffness/film stiffness.}$$

- Treat substrate as semi-infinite isotropic elastic halfspace.
- Given surface displacement (w_1, w_2, u_3), solve 3D linear elasticity problem in substrate by separation of variables.
- Substrate energy is the result (modulo constants).

Total energy = membrane + bending + substrate

To permit spatial averaging, we assume periodicity on some (large) scale L, and we focus on the energy per unit area:

$$
\begin{aligned}
E_h \quad = \quad & \frac{\alpha_m h}{L^2} \int_{[0,L]^2} \left| e(w) + \tfrac{1}{2}\nabla u_3 \otimes \nabla u_3 - \eta I \right|^2 dx\, dy && (\textit{membrane}) \\
& + \frac{h^3}{L^2} \int_{[0,L]^2} |\nabla\nabla u_3|^2\, dx\, dy && (\textit{bending}) \\
& + \frac{\alpha_s}{L^2} \left(\|w\|^2_{H^{1/2}} + \|u_3\|^2_{H^{1/2}} \right) && (\textit{substrate})
\end{aligned}
$$

where h is the thickness of the film.

- We have already normalized by stiffness of the film, so α_m, α_s, η are dimensionless parameters:
  - α_m (order 1) comes from mechanics of bending;
  - α_s (small) is the ratio (substrate stiffness)/(film stiffness);
  - η (small, pos) is the misfit.
- Unwrinkled state $(w_1, w_2, u_3) = 0$ has energy $\alpha_m \eta^2 h$.

The energy scaling law

Theorem

If h/L and η are small enough, the minimum energy satisfies

$$\min E_h \sim \min\{\alpha_m \eta^2 h, \alpha_s^{2/3} \eta h\};$$

moreover

- *the first alternative corresponds to the unwrinkled state; it is better when $\alpha_m \eta < \alpha_s^{2/3}$.*
- *the second alternative is achieved by a herringbone pattern using wrinkles with length scale $\alpha_s^{-1/3} h$, whose direction oscillates on a suitable scale (longer but not fully determined).*

The smallness conditions are explicit:

$$\alpha_m \alpha_s^{-4/3} (h/L)^2 \leq 1 \quad \text{and} \quad \eta^2 \leq \alpha_m^{-1} \alpha_s^{2/3}.$$

Perhaps other, less-ordered patterns could also be optimal (e.g. “labyrinths”).

The energy scaling law – cont'd

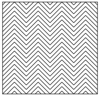
$$\min E_h \sim \min\{\alpha_m \eta^2 h, \alpha_s^{2/3} \eta h\};$$

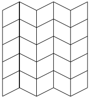
One consequence: the Miura-ori pattern is not optimal:

- Its scaling law is $\alpha_m^{1/6} \alpha_s^{5/8} \eta^{17/16} h$.
- If film prefers not to be flat ($\alpha_m \eta \gg \alpha_s^{2/3}$) then Miura-ori energy $\gg$ herringbone energy.

Intuition:

- Bending energy requires folds of Miura-ori pattern to be rounded.
- Where folds intersect this costs significant membrane energy.
- In herringbone pattern the membrane term isn't identically zero, but it does not contribute at leading order.





Tasks for analysis

Our assertion is:

$$\begin{aligned} \min E_h &\geq C_1 \min\{\alpha_m \eta^2 h, \alpha_s^{2/3} \eta h\} \text{ and} \\ \min E_h &\leq C_2 \min\{\alpha_m \eta^2 h, \alpha_s^{2/3} \eta h\} \end{aligned}$$

with C_1, C_2 independent of h, α_s, and α_m, provided

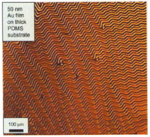
$$\alpha_m \alpha_s^{-4/3} (h/L)^2 \leq 1 \quad \text{and} \quad \eta^2 \leq \alpha_m^{-1} \alpha_s^{2/3}.$$

Our tasks are thus

- to prove an upper bound, by describing and optimizing the herringbone pattern; and
- to prove a "matching" ansatz-free lower bound.

The upper bound – overview

- Energy of unbuckled state is $\alpha_m \eta^2 h$
- Energy of herringbone is $C\alpha_s^{2/3}\eta h$
- So min $E_h \leq \min\{\alpha_m \eta^2 h, C\alpha_s^{2/3}\eta h\}$



$$
\begin{aligned}
E_h \quad = \quad & \frac{\alpha_m h}{L^2} \int_{[0,L]^2} |e(w) + \tfrac{1}{2}\nabla u_3 \otimes \nabla u_3 - \eta I|^2 \, dx\, dy \\
& + \frac{h^3}{L^2} \int_{[0,L]^2} |\nabla\nabla u_3|^2 \, dx\, dy + \frac{\alpha_s}{L^2} \left(\|w\|^2_{H^{1/2}} + \|u_3\|^2_{H^{1/2}} \right)
\end{aligned}
$$

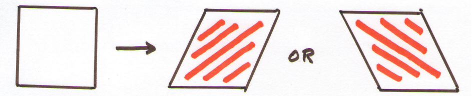
Key features of herringbone:

- membrane term is negligible
- typical slope is $|\nabla u_3| \sim \sqrt{\eta}$
- typical in-plane strain is $|e(w)| \sim \eta$ (smaller!)
- scale of wrinkling set by competition between bending term and u_3 part of substrate term
- two types of wrinkling must mix for $e(w)$ to have average 0; but the longer length scale is not fully determined.

More detail on the herringbone pattern

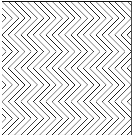
Film wants to expand (isotropically) relative to substrate.

- 1D wrinkling expands only transverse to the wrinkles
- a simple shear expands one diag dirn, compresses the other
- shear combined with wrinkling achieves isotropic expansion



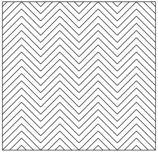
Substrate prohibits large deformation; therefore the film mixes the two shear-combined-with-wrinkling variants. Thus, the herringbone pattern has two length scales:

- The smaller one (the scale of the wrinkling) is set by competition between *bending* term and *substrate energy of* u_3.
- The larger one (scale of the phase mixture) must be s.t. the *substrate energy of w is insignificant*. (It is not fully determined.)



Another perspective

- Mixture of two symmetry-related "phases"
- "Phase 1" uses sinusoidal wrinkles perp to $(1, 1)$, superimposed on an in-plane shear.
- "Phase 2" uses wrinkles perp to $(1, -1)$, superimposed on a different shear.



In phase 1: $e(w) + \frac{1}{2}\nabla u_3 \otimes \nabla u_3 = \begin{pmatrix} 0 & -\eta \\ -\eta & 0 \end{pmatrix} + \begin{pmatrix} \eta & \eta \\ \eta & \eta \end{pmatrix} = \eta I;$

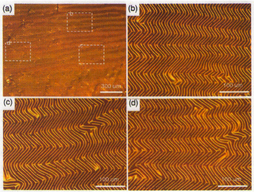
In phase 2: $e(w) + \frac{1}{2}\nabla u_3 \otimes \nabla u_3 = \begin{pmatrix} 0 & \eta \\ \eta & 0 \end{pmatrix} + \begin{pmatrix} \eta & -\eta \\ -\eta & \eta \end{pmatrix} = \eta I;$

Membrane term vanishes (except in transition layers between the phases)! Since avg in-plane shear is 0, in-plane displacement w can be periodic.

Scale of shear oscillation can be much longer than scale of wrinkling, since $e(w) \sim \eta$ while $u_3 \sim \sqrt{\eta}$, and $\eta \ll 1$.

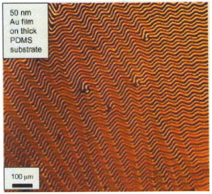
Scale of shear oscillation must be small enough: substrate energy of shear osc $\leq$ substrate energy of wrinkling.

Scale of shear oscillation must be large enough: membrane energy of transition layers should be $\leq$ other energy assoc with wrinkling.



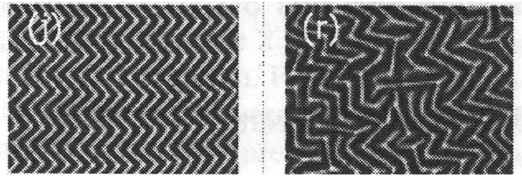
silicon on pdms

Song et al, *J Appl Phys* 103 (2008) 014303



gold on pdms

Chen & Hutchinson, *Scripta Mat* 50 (2004) 797–801



different release histories

Lin & Yang, *Appl Phys Lett* 90 (2007) 241903

No other pattern can do better

Want to show: For any periodic (w_1, w_2, u_3), $E_h \geq C \min\{\alpha_m \eta^2 h, \alpha_s^{2/3} \eta h\}$.

The proof is surprisingly easy. To simplify notation, take the period to be $L = 1$. We'll use only that

$$\text{membrane term} \geq \alpha_m h \int |\partial_1 w_1 + \tfrac{1}{2}|\partial_1 u_3|^2 - \eta|^2 \, dx\, dy,$$

$$\text{bending term} = h^3 \|\nabla\nabla u_3\|_{L^2}^2, \quad \text{and} \quad \text{substrate term} \geq \alpha_s \|u_3\|_{H^{1/2}}^2.$$

CASE 1, FIRST PASS: If $\int (\partial_1 u_3)^2 \ll \eta$ then membrane $\gtrsim \alpha_m \eta^2 h$, since $\partial_x w_1$ has mean 0.

CASE 2, FIRST PASS: If $\int (\partial_1 u_3)^2 \gtrsim \eta$ use the interpolation inequality

$$\|\nabla u_3\|_{L^2} \lesssim \|\nabla\nabla u_3\|_{L^2}^{1/3} \|u_3\|_{H^{1/2}}^{2/3}$$

to see that

$$\begin{aligned} \text{Bending + substrate terms} &= h^3\|\nabla\nabla u_3\|^2 + \tfrac{1}{2}\alpha_s\|u_3\|_{H^{1/2}}^2 + \tfrac{1}{2}\alpha_s\|u_3\|_{H^{1/2}}^2 \\ &\gtrsim \left(h^3\|\nabla\nabla u_3\|^2 \alpha_s^2 \|u_3\|_{H^{1/2}}^4\right)^{1/3} \\ &\gtrsim h\alpha_s^{2/3}\|\nabla u_3\|_{L^2}^2 \gtrsim h\alpha_s^{2/3}\eta \end{aligned}$$

No other pattern can do better

Want to show: For any periodic (w_1, w_2, u_3), $E_h \geq C \min\{\alpha_m \eta^2 h, \alpha_s^{2/3} \eta h\}$.

The proof is surprisingly easy. To simplify notation, take the period to be $L = 1$. We'll use only that

$$\text{membrane term} \geq \alpha_m h \int |\partial_1 w_1 + \tfrac{1}{2}|\partial_1 u_3|^2 - \eta|^2 \, dx\, dy,$$

$$\text{bending term} = h^3 \|\nabla\nabla u_3\|_{L^2}^2, \quad \text{and} \quad \text{substrate term} \geq \alpha_s \|u_3\|_{H^{1/2}}^2.$$

CASE 1, FIRST PASS: If $\int (\partial_1 u_3)^2 \ll \eta$ then membrane $\gtrsim \alpha_m \eta^2 h$, since $\partial_x w_1$ has mean 0.

CASE 2, FIRST PASS: If $\int (\partial_1 u_3)^2 \gtrsim \eta$ use the interpolation inequality

$$\|\nabla u_3\|_{L^2} \lesssim \|\nabla\nabla u_3\|_{L^2}^{1/3} \|u_3\|_{H^{1/2}}^{2/3}$$

to see that

$$\begin{aligned}
\text{Bending + substrate terms} &= h^3 \|\nabla\nabla u_3\|^2 + \tfrac{1}{2}\alpha_s \|u_3\|_{H^{1/2}}^2 + \tfrac{1}{2}\alpha_s \|u_3\|_{H^{1/2}}^2 \\
&\gtrsim \left(h^3 \|\nabla\nabla u_3\|^2 \alpha_s^2 \|u_3\|_{H^{1/2}}^4\right)^{1/3} \\
&\gtrsim h\alpha_s^{2/3} \|\nabla u_3\|_{L^2}^2 \gtrsim h\alpha_s^{2/3} \eta
\end{aligned}$$

No other pattern can do better

Want to show: For any periodic (w_1, w_2, u_3), $E_h \geq C \min\{\alpha_m \eta^2 h, \alpha_s^{2/3} \eta h\}$.

The proof is surprisingly easy. To simplify notation, take the period to be $L = 1$. We'll use only that

$$\text{membrane term} \geq \alpha_m h \int |\partial_1 w_1 + \tfrac{1}{2}|\partial_1 u_3|^2 - \eta|^2 \, dx \, dy,$$

$$\text{bending term} = h^3 \|\nabla\nabla u_3\|_{L^2}^2, \quad \text{and} \quad \text{substrate term} \geq \alpha_s \|u_3\|_{H^{1/2}}^2.$$

CASE 1, FIRST PASS: If $\int (\partial_1 u_3)^2 \ll \eta$ then membrane $\gtrsim \alpha_m \eta^2 h$, since $\partial_x w_1$ has mean 0.

CASE 2, FIRST PASS: If $\int (\partial_1 u_3)^2 \gtrsim \eta$ use the interpolation inequality

$$\|\nabla u_3\|_{L^2} \lesssim \|\nabla\nabla u_3\|_{L^2}^{1/3} \|u_3\|_{H^{1/2}}^{2/3}$$

to see that

$$\begin{aligned} \text{Bending + substrate terms} &= h^3 \|\nabla\nabla u_3\|^2 + \tfrac{1}{2}\alpha_s \|u_3\|_{H^{1/2}}^2 + \tfrac{1}{2}\alpha_s \|u_3\|_{H^{1/2}}^2 \\ &\gtrsim \left(h^3 \|\nabla\nabla u_3\|^2 \alpha_s^2 \|u_3\|_{H^{1/2}}^4 \right)^{1/3} \\ &\gtrsim h\alpha_s^{2/3} \|\nabla u_3\|_{L^2}^2 \gtrsim h\alpha_s^{2/3} \eta \end{aligned}$$

More detail on case 1

CASE 1, SECOND PASS: Suppose $\int \frac{1}{2}(\partial_1 u_3)^2 \leq \frac{1}{2}\eta$. Use the inequality

$$\int \partial_1 w_1 + \tfrac{1}{2}(\partial_1 u_3)^2 - \eta \leq \left(\int |\partial_1 w_1 + \tfrac{1}{2}(\partial_1 u_3)^2 - \eta|^2 \right)^{1/2}$$

(recall that we took $L = 1$ to simplify the notation). Since w_1 is periodic,

$$\text{LHS} = \int \tfrac{1}{2}(\partial_1 u_3)^2 - \eta \geq \tfrac{1}{2}\eta.$$

So

$$h\alpha_m \int |\partial_1 w_1 + \tfrac{1}{2}(\partial_1 u_3)^2 - \eta|^2 \geq h\alpha_m(\tfrac{1}{2}\eta)^2$$

More detail on case 2

CASE 2, SECOND PASS: Suppose $\int \frac{1}{2}|\partial_1 u_3|^2 \geq \frac{1}{2}\eta$. Then evidently

$$\int |\nabla u_3|^2 \geq \eta.$$

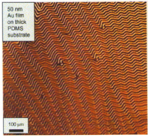
Our first pass argument combined the inequality $\frac{a+b+c}{3} \geq (abc)^{1/3}$ with an interpolation inequality, which can be written

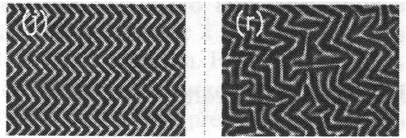
$$\int |\nabla u|^2 \leq \left(\int |\nabla\nabla u|^2\right)^{1/3} \left(\int |\nabla^{1/2} u|^2\right)^{2/3}$$

using the suggestive notation $\sum |k||\hat{u}(k)|^2 = \int |\nabla^{1/2} u|^2$. Proof of this ineq is easy in Fourier space:

$$\begin{aligned} \sum |k|^2 |\hat{u}(k)|^2 &= \sum |k|^{4/3} |\hat{u}(k)|^{2/3} \cdot |k|^{2/3} |\hat{u}(k)|^{4/3} \\ &\leq \left(\sum |k|^4 |\hat{u}(k)|^2\right)^{1/3} \left(\sum |k||\hat{u}(k)|^2\right)^{2/3} \end{aligned}$$

Stepping back





Main accomplishment: scaling law of the minimum energy, based on

- upper bound, corresponding to the herringbone pattern, and
- lower bound, using little more than interpolation.
- Key point: they agree (up to a factor indep of h, η, and α_s).

Open question: what about those labyrinth patterns?

- Why are they seen in some numerical and physical experiments (but not in others)? Do they achieve the same scaling law, or are they higher-energy local minima?

A closely related problem

What if the film can relieve the misfit by blistering?

FeNi on salt
(from Gioia & Ortiz, 1997)

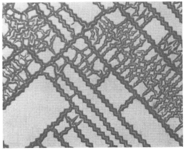


FIG. 7. Regular web of telephone cords in a 75–25 Fe–Ni film grown by epitaxial evaporation on rock salt. The ⟨001⟩ orientation of the cords is clearly apparent except near steps in the NaCl. After Yelon and Voegeli (1964). Reprinted with permission of Pergamon Press.

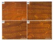
Very different from perfectly-bonded case, since substrate feels only in-plane displacement of bonded region.

Recent joint work with Jacob Bedrossian (CPAM in press, and preprint at arxiv): there is a regime where a lattice-like blistering pattern is energetically preferred over a few large blisters (if the area fraction of blistering is fixed).

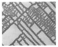
Image credits

P.-C. Lin & S. Yang, *Appl Phys Lett* 90 (2007) 241903

X. Chen and J. Hutchinson, *Scripta Materialia* 50 (2004) 797–801



J. Song et al, *J Appl Phys* 103 (2008) 014303



G. Gioia & M. Ortiz, *Adv Appl Mech* 33 (1997) 119-192