Wrinkling of thin elastic sheets – Lecture 5: The floating elastic sheet

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PCMI, July 2014

Orientation

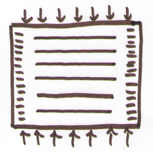
- Recall: *tension-induced* and *compressive* wrinkling are rather different. The annulus problem (Lecture 3) was tension-driven; the herringbone problem (Lecture 4) was compressive.
- Also recall the classic model problem

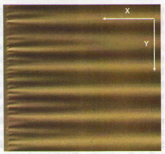
  $$\min_{\substack{u=0 \text{ at } x=0 \\ u_y=\pm 1}} \int u_x^2 + \varepsilon |u_{yy}|$$

  where the *length scale* of the microstructure *depends on dist to bdry* (Lecture 2, and problem 3 of TA Session 2).
- Today's discussion concerns an example of tension-induced wrinkling that's analogous to that model problem.
- Joint work with Hoai-Minh Nguyen (preprint to come soon).

The floating elastic sheet

- sheet floats on water
- confined on 2 sides
- surface tension pulls free edges
- wrinkles form, refining at free edges





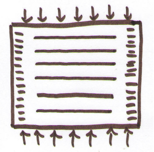
Experiment and some theory: Huang, Davidovitch, Santangelo, Russell, Menon (PRL 2010); also Davidovitch (PRE 2009)

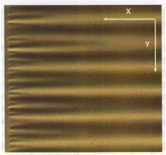
Focus of today's discussion:

- Identify analogue of the relaxed problem and its solution.
- Estimate the energetic cost of changing the wrinkling length scale.

Heuristics

- sheet floats on water
- confined on 2 sides
- surface tension pulls free edges
- wrinkles form, refining at free edges





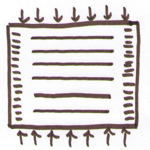
In the center: If length is large enough, end effects become irrelevant near center. Confinement requires wrinkling. Surface tension makes it tension-driven. Scale set by gravitational effects (prefers small amplitude) and bending (prefers few wrinkles).

At ends: Surface tension (miniscus effects) demand much smaller length scale.

Near ends: Refinement of wrinkling scale costs energy. A major goal of today's discussion: how to quantify this.

The energy

- Von Karman theory: displacement (w_1, w_2, u_3)
- Confinement: $w_2(x,0) = 0$, $w_2(x,1) = -\Delta/2$
- Domain $\Omega = [0,L] \times [0,1]$, with $L \geq 1$
- For simplicity: u_3 is periodic in y



$$
\begin{aligned}
E_h \quad = \quad & \alpha_m h \int_\Omega |e(w) + \tfrac{1}{2}\nabla u_3 \otimes \nabla u_3|^2 + h^3 \int_\Omega |\nabla\nabla u_3|^2 \\
& + \alpha_g \int_\Omega u_3^2 + \tfrac{1}{2}\alpha_s \int_\Omega |\nabla u_3|^2 - (\alpha_c - \alpha_s) \int_\Omega \partial_x w_1 + \alpha_c \|u_3\|^2_{H^{1/2}(\Gamma)}
\end{aligned}
$$

Key hypothesis: $\alpha_c > \alpha_s$. Notation: Γ = free edges. Notes:

- Extra surf energy due to $u_3 \neq 0$ is $\alpha_s \int_\Omega \frac{1}{2}|\nabla u_3|^2 + \alpha_c \|u_3\|^2_{H^{1/2}(\Gamma)}$
- Extra surf energy due to in-plane def is $(\alpha_s - \alpha_c) \int_\Omega \operatorname{div} w$
- Since $\int_\Omega \operatorname{div} w = \int_\Omega \partial_x w_1 + \text{const}$, surf tension is tensile when $\alpha_c > \alpha_s$.
- $H^{1/2}$ term is energy of capillary fringe field:

$$
\|u_3\|^2_{H^{1/2}(\Gamma)} = \min \int_{\text{water}} |\nabla u_3|^2 = \sum_k |k| \left(|\hat{u}_3(0,k)|^2 + |\hat{u}_3(L,k)|^2 \right)
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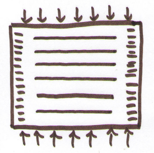
The energy



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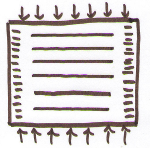

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Parameters, conventions, expectations

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\end{aligned}
$$

α_m	dimensionless
α_g	dimension: 1/length; gravitation, normalized by film stiffness
α_s	dimension: length; water-sheet surface energy, normalized
α_c	dimension: length; water-air surface energy, normalized
u_3	mean 0, since water is incompressible

We expect

$$
\min E_h = \begin{pmatrix} \text{value assoc} \\ \text{tensile effects} \end{pmatrix} + \begin{pmatrix} \text{correction due to presence} \\ \text{of bending energy} \end{pmatrix}.
$$

First term is like "relaxed energy," second term is like "excess energy due to positive h," except that we haven't nondimensionalized.

A convenient reorganization

$$E_h = E_1 + E_2 + E_3 + E_4$$

$$\begin{aligned}
E_1 &= \alpha_m h \int_\Omega (\partial_x w_1 + \tfrac{1}{2}|\partial_x u_3|^2)^2 - (\alpha_c - \alpha_s) \int_\Omega (\partial_x w_1 + \tfrac{1}{2}|\partial_x u_3|^2) \\
E_2 &= \ \alpha_m h \int_\Omega (\partial_y w_2 + \tfrac{1}{2}|\partial_y u_3|^2)^2 + \tfrac{1}{2}\alpha_s \int_\Omega |\partial_y u_3|^2 \\
E_3 &= h^3 \int_\Omega |\nabla\nabla u_3|^2 + \tfrac{1}{2}\alpha_c \int_\Omega |\partial_x u_3|^2 + \alpha_g \int_\Omega u_3^2 + \alpha_c \|u_3\|^2_{H^{1/2}(\Gamma)} \\
E_4 &= \tfrac{1}{2}\alpha_m h \int_\Omega |\partial_x w_2 + \partial_y w_1 + \partial_x u_3 \partial_y u_3|^2
\end{aligned}$$

- E_1 captures stretching due to surface tension. It slaves $\partial_1 w_1$ to $\partial_x u_3$.
- E_2 captures effect of confining bdry conditions. It determines $\int_0^1 (\partial_y u_3)^2$.
- Minimization of E_1 and E_2 gives the “value assoc to tensile forces.”
- E_3 captures effect of bending resistance. Its min value is the “correction due to bending energy.”
- E_4 is unimportant due to symmetry of bdry conditions.

Energy due to tensile forces

$$E_1 = \alpha_m h \int_\Omega (\partial_x w_1 + \tfrac{1}{2}|\partial_x u_3|^2)^2 - (\alpha_c - \alpha_s) \int_\Omega (\partial_x w_1 + \tfrac{1}{2}|\partial_x u_3|^2)$$

$$E_2 = \alpha_m h \int_\Omega (\partial_y w_2 + \tfrac{1}{2}|\partial_y u_3|^2)^2 + \tfrac{1}{2}\alpha_s \int_\Omega |\partial_y u_3|^2$$

- E_1 captures stretching due to surface tension. To minimize integrand, E_1 prefers $\partial_x w_1 + \frac{1}{2}|\partial_x u_3|^2$ to be a special value. So $\partial_x w_1$ is slaved to $\partial_x u_3$.
- E_2 captures effect of the confining bdry condition. Bdry condn gives $\int_0^1 \partial_y w_2 \, dy = -\Delta/2$. Claim: E_2 is minimized when, at each x, $z = \int_0^1 |\partial_y u_3|^2 \, dy$ achieves

$$\min_{z>0} \frac{\alpha_m h}{4}(\Delta - z)^2 + \frac{1}{2}\alpha_s z.$$

- No wrinkling if best z is 0 (this occurs when $\alpha_m h \Delta < \alpha_s$).

Analysis of E_2

$$E_2 = \alpha_m h \int_\Omega (\partial_y w_2 + \tfrac{1}{2}|\partial_y u_3|^2)^2 + \tfrac{1}{2}\alpha_s \int_\Omega |\partial_y u_3|^2$$

For any x, let $z = z(x) = \int_0^1 (\partial_y u_3)^2 \, dy$. We have

$$\begin{aligned} -\tfrac{1}{2}\Delta + \tfrac{1}{2}z &= \int_0^1 \partial_y w_2 + \tfrac{1}{2}(\partial_y u_3)^2 \, dy \\ &\leq \left(\int_0^1 |\partial_y w_2 + \tfrac{1}{2}(\partial_y u_3)^2|^2 \right)^{1/2} . \end{aligned}$$

Therefore

$$\begin{aligned} E_2 &\geq \int_0^L \alpha_m h \left(\frac{z(x) - \Delta}{2} \right)^2 + \tfrac{1}{2}\alpha_s z(x) \\ &\geq L \min_{z \geq 0} \left(\alpha_m h \left(\frac{z - \Delta}{2} \right)^2 + \tfrac{1}{2}\alpha_s z \right) . \end{aligned}$$

Moreover: if $\int_0^1 |\partial_y u_3|^2 \, dy$ deviates from the optimal value, E_2 will be larger (by an amount that's easy to determine).

Summary thus far

The picture so far:

$$E_1 \geq -\frac{(\alpha_c - \alpha_s)^2}{4\alpha_m h} L, \quad E_2 \geq -\frac{1}{4}\alpha_m h \Delta^2 L - \frac{[\alpha_m h\Delta - \alpha_s]_+^2}{4\alpha_m h} L.$$

Equality holds in former when $\partial_x w_1 + \frac{1}{2}|\partial_x u_3|^2$ has the preferred value, so

$\partial_x w_1$ is slaved to $\partial_x u_3$.

Equality holds in the latter when

$$\int_0^1 (\partial_y u_3)^2 \, dy = \delta \text{ for all } x,$$

with

$$\delta = \left[\frac{\alpha_m h\Delta - \alpha_s}{\alpha_m h}\right]_+$$

(which should be positive, to see wrinkling).

$\min E_1 + \min E_2$ is analogous to the min “relaxed energy” $\mathcal{E}_0$ of Lectures 2 & 3. If “excess energy” is small, then conditions noted above must be approx met.

Correction due to the bending term

Moving forward: E_3 controls the wrinkling. Recall:

$$E_3 = h^3 \int_\Omega |\nabla\nabla u_3|^2 + \tfrac{1}{2}\alpha_c \int_\Omega |\partial_x u_3|^2 + \alpha_g \int_\Omega u_3^2 + \alpha_c \|u_3\|^2_{H^{1/2}(\Gamma)}.$$

- The capillary term forces u_3 to be small at unconfined edges $x = 0, L$.
- If u_3 is uniformly small, then length scale of wrinkling must be small (since avg of $|\partial_y u_3|^2$ was fixed by E_2). Expensive wrt bending energy.
- If u_3 is not uniformly small, then $\partial_x u_3$ must be large. Expensive wrt surface energy.

Evidently: length scale of wrinkling should vary with distance to edge. Focus for discussion: how does it vary, and what is the associated excess energy?

A heuristic argument

Arguing as for refinement of wrinkles in "annulus problem," we can construct $u_3(x, y)$ st $\int_0^1 (\partial_y u_3)^2\, dy = \delta$ and there's a local length scale $\ell(x)$ for the wrinkling, with

$$|u_3| \sim \delta^{1/2}\ell(x), \quad |\partial_x u_3| \sim \delta^{1/2}\ell'(x) \quad |\nabla\nabla u_3| \sim \delta^{1/2}\ell^{-1}(x).$$

Since $E_3 = h^3 \int_\Omega |\nabla\nabla u_3|^2 + \frac{1}{2}\alpha_c \int_\Omega |\partial_x u_3|^2 + \alpha_g \int_\Omega u_3^2 + \alpha_c \|u_3\|^2_{H^{1/2}(\Gamma)}$ we get

$$E_3 \sim h^3 \int_0^L \delta\ell^{-2}\, dx + \tfrac{1}{2}\alpha_c \int_0^L \delta|\ell'|^2\, dx + \alpha_g \int_0^L \ell^2\, dx + \alpha_c\delta(\ell(0) + \ell(L)).$$

Note: the final term is linear in ℓ since we expect

$$\int_0^1 |\partial_y^{1/2} u_3|^2\, dy \sim \left(\int_0^1 u_3^2\, dy\right)^{1/2} \left(\int_0^1 |\partial_y u_3|^2\, dy\right)^{1/2}.$$

Variation of the local length scale

Conclusion thus far: wrinkling length scale should minimize

$$\int_0^L \left(h^3 \ell^{-2} + \tfrac{1}{2}\alpha_c |\ell'|^2 + \alpha_g \ell^2 \right) \, dx + \alpha_c(\ell(0) + \ell(L)).$$

What to do with this?

(a) Study this 1D variational problem for $\ell(x)$.

(b) Draw qualitative conclusions, then seek ansatz-free results for E_3.

I'll focus on (b).

Easy warmup. If L is large enough, edge effects should be unimportant and ℓ should be indep of x. The assoc *bulk wrinkling scale* ℓ_B should achieve

$$\min_{\ell} h^3 \ell^{-2} + \alpha_g \ell^2$$

whence $\ell_B = (h^3 \alpha_g^{-1})^{1/4}$. We know the ansatz-free lower bound associated with this calculation: it uses the interpolation ineq

$$\int_0^1 |\partial_y u_3|^2 \, dy \le \left(\int_0^1 u_3^2 \, dy \right)^{1/2} \left(\int_0^1 |\partial_{yy} u_3|^2 \, dy \right)^{1/2}.$$

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The energetic cost of refinement

A different question: Suppose $\ell(0) \ll \ell_B$, and L is large enough that ℓ_B is achieved at $x = L/2$. What is the energetic cost of refinement?

In terms of ℓ the answer is easy:

$$\int_0^{L/2} h^3 \ell^{-2} + \tfrac{1}{2}\alpha_c |\ell'|^2 \, dx \geq \int_0^{L/2} \sqrt{2} h^{3/2} \alpha_c^{1/2} |\ell'/\ell| \, dx$$
$$\geq \sqrt{2} h^{3/2} \alpha_c^{1/2} \ln(\ell_B/\ell(0)).$$

Does this have an ansatz-free analogue? Yes! It is provided (after nondimensionalization) by

Proposition: If $g(x, y) : [0, 1]^2 \to \mathbb{R}$ is periodic in y and

- $\int_0^1 g^2(0, y)\, dy \leq a$ (sufficiently small)
- $\int_0^1 g^2(1, y)\, dy \geq b$ (sufficiently large)
- $\int_0^1 (\partial_y g)^2(x, y)\, dy \geq 1$ for most x (all but measure ε)

then

$$t \int_0^1 \int_0^1 (\partial_{yy} g)^2 + \alpha \int_0^1 \int_0^1 (\partial_x g)^2 \geq C\sqrt{\alpha t} \log(b/a)$$

provided $\varepsilon \leq \frac{\alpha\sqrt{ab}}{2\sqrt{\alpha t}\log(b/a)}$.

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Proof of the proposition

Sketch of proof, assuming for simplicity that $\int_0^1 g^2(0,y)\,dy = a$, $\int_0^1 g^2(1,y) = b$, and $b = 4^n a$ for some n.

Step 1: Let x_j be the largest x st $\int_0^1 g^2(x,y)\,dy \geq 4^j a$. Note that $0 \leq x_0 < x_1 < \cdots < x_n = 1$ and

$$\int_0^1 g^2(x,y)\,dy < 4^j a \quad \text{for } x > x_j;$$

also $\int_0^1 g^2(x_j,y)\,dy = 4^j a$, and (by triangle ineq)

$$\left(\int_0^1 |g(x_j,y) - g(x_{j+1},y)|^2\,dy\right)^{1/2} \geq \left(\int_0^1 g^2(x_{j+1},y)\,dy\right)^{1/2} - \left(\int_0^1 g^2(x_j,y)\,dy\right)^{1/2}$$

Step 2: Observe that

$$\int_{\substack{x_j<x<x_{j+1}\\0<y<1}} (\partial_x g)^2\,dx\,dy \geq \frac{1}{|x_{j+1}-x_j|}\int_0^1 |g(x_j,y) - g(x_{j+1},y)|^2\,dy$$

since LHS is minimized by the function that's affine in x with the same values at $x = x_j$ and x_{j+1}.

Proof of the proposition, cont'd

Step 3: Suppose now that $\int_0^1 g_y^2 \, dy \geq 1$ on at least half of (x_j, x_{j+1}). Then our much-used interpolation ineq $\left(\int |\partial_y g|^2 \, dy\right)^2 \leq \int g^2 \, dy \cdot \int |\partial_{yy} g|^2 \, dy$ gives

$$\int_{\substack{x_j < x < x_{j+1} \\ 0 < y < 1}} g_{yy}^2 \, dx \, dy \geq \frac{1}{4^j a} \frac{1}{2} |x_{j+1} - x_j|.$$

Combining this with outcome of step 2, we get

$$\begin{aligned} t \int_{\substack{x_j < x < x_{j+1} \\ 0 < y < 1}} g_{yy}^2 \, dx \, dy \quad &+ \quad \alpha \int_{\substack{x_j < x < x_{j+1} \\ 0 < y < 1}} g_x^2 \, dx \, dy \\ &\geq \quad \frac{t}{4^j a} \cdot \tfrac{1}{2} |x_{j+1} - x_j| + \frac{\alpha}{|x_{j+1} - x_j|} \cdot 4^j a \\ &\geq \quad C \sqrt{\alpha t}. \end{aligned}$$

Repeating for each j, we get a total of order $\sqrt{\alpha t} \log(b/a)$ since $n \sim \log(b/a)$.

Proof of the proposition – cont'd

Step 4. We assumed in Step 3 that $\int_0^1 g_y^2\,dy \geq 1$ on at least half of (x_j, x_{j+1}). It's sufficient for Step 3 that this hold for at least half those intervals, eg for $j \geq n/2$.

Suppose Step 3 fails, because this condition fails for some $j \geq n/2$. Then

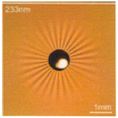
$$|x_j - x_{j+1}| \leq C\varepsilon \leq C\frac{\alpha\sqrt{ab}}{2\sqrt{\alpha t}\log(b/a)}$$

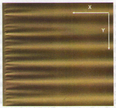
and $4^j \geq (b/a)^{1/2}$. In this case

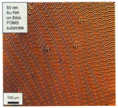
$$\begin{aligned}\alpha\int_{\substack{x_j<x<x_{j+1}\\0<y<1}} g_x^2\,dx\,dy &\geq \frac{\alpha}{|x_{j+1}-x_j|}4^j a\\ &\geq \frac{\alpha\sqrt{ab}}{|x_{j+1}-x_j|} \geq C\sqrt{\alpha t}\log(b/a)\end{aligned}$$

and the estimate is true anyway.

Stepping back

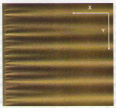


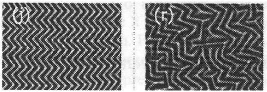


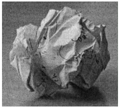


- Wrinkling can be microstructure (when its length scale $\to 0$ as $h \to 0$).
- Advantage of focus on energy scaling law: permits ansatz-free analysis.
- Different tools have different strengths:
  - Numerics predicts patterns, but finds local minima
  - Minimization within an ansatz suggests what to expect
  - Ansatz-independent lower bounds confirm (or refute) adequacy of a particular ansatz
  - Lower bounds help explain *why* patterns form, and which features are most important

Stepping back – cont'd



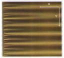




- Some successes, but still many open problems.
  - For the floating elastic sheet: is the local scale of wrinkling correctly predicted by our 1D var'l problem for $\ell(x)$?
  - For the film bonded to a compliant substrate, are the labyrinth patterns comparable energetically to the herringbone pattern?
  - For a "crumpled" piece of paper, does energy minimization require folds? How does the min energy scale with h?
- Wrinkling is just one example of energy-driven pattern formation
  - Shape-memory materials, ferromagnets, liquid crystals, ...
  - Energy min is a good guide in some systems, not in others.
  - Many examples are "understood," through numerics or minimization within an ansatz.
  - Attempts at rigorous analysis challenge our understanding.

Credits

Images are from:



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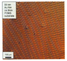

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