PCMI – Kohn – Problems for TA Session 2

Problems 1 and 2 are related to the relaxation of our membrane energies (discussed in Lecture 2). Problem 3 (which is independent of the others, and could easily occupy the entire hour-long TA session) develops the most accessible aspects of a classic example where the length scale of the microstructure varies with position.

(1) In Lecture 2 we discussed the relaxation of the membrane energy $W(Dg) = |Dg^T Dg - I|^2$, which can also be written as $W(Dg) = (\lambda_1^2 - 1)^2 + (\lambda_2^2 - 1)^2$ where λ_1 and λ_2 are the principal stretches (the eigenvalues of $(Dg^T Dg)^{1/2}$). The relaxation turned out to be $f(Dg) = (\lambda_1^2 - 1)_+^2 + (\lambda_2^2 - 1)_+^2$. In the argument showing that the relaxation is less than or equal to f, we made repeated use of "one-dimensional oscillations." That argument can be generalized as follows:

LAYERING LEMMA: *Let M_0 and M_1 be $m \times n$ matrices, with the property that $M_1 - M_0 = a \otimes n$ for some $a \in R^m$ and $n \in R^n$. Fixing θ between 0 and 1, let $M_\theta = (1-\theta)M_0 + \theta M_1$. Then for any (reasonable, bounded) domain $D \subset R^n$ and any $\varepsilon > 0$, there is a function $g_\varepsilon : D \to R^m$ such that*

- *$g_\varepsilon(x) = M_\theta \cdot x$ for $x \in \partial D$,*
- *the subset of D where $Dg_\varepsilon = M_1$ has measure $\theta|D| \pm C\varepsilon$,*
- *the subset of D where $Dg_\varepsilon = M_0$ has measure $(1-\theta)|D| \pm C\varepsilon$,*
- *the maximum gradient is controlled: $\|Dg_\varepsilon\|_{L^\infty(D)} \le C$.*

Here C is a positive constant that's independent of ε.

OK, here are the questions:

(a) Identify the sense in which the Layering Lemma generalizes what we did in Lecture 2.

(b) Prove the special case of the Layering Lemma that was used in Lecture 2.

(c) Once you have done (b), proving the Layering Lemma in full generality is not much more difficult. Try it.

(2) In Lecture 2, I asserted that the relaxation of the von Karman membrane energy $W = |e(w) + \frac{1}{2}\nabla u_3 \otimes \nabla u_3|^2$ is $f = (\mu_1)_+^2 + (\mu_2)_+^2$, where μ_i are the eigenvalues of $e(w) + \frac{1}{2}\nabla u_3 \otimes \nabla u_3$ and $(t)_+ = \max\{t, 0\}$.

(a) Using an argument parallel to that applied in the nonlinear setting, show that the relaxed energy is less than or equal to f.

(b) Using an argument parallel to that applied in the nonlinear setting, show that the relaxed energy is greater than or equal to f.

(c) Show that f is a convex function of e and ∇u, by fillling in the details of the following argument:

(i) We are working with the function $W(e,\xi) = |e + \frac{1}{2}\xi \otimes \xi|^2$, defined on symmetric 2×2 matrices e and 2-vectors ξ. Check that the relaxation $f(e,\xi) = (\mu_1)_+^2 + (\mu_2)_+^2$ has the alternative expression

$$f(e,\xi) = \min\{|e + \tfrac{1}{2}\xi \otimes \xi + M|^2 \,:\, M \geq 0\} \tag{1}$$

in which M ranges over 2×2 symmetric matrices with nonnegative eigenvalues.

(ii) Fix e_0, ξ_0, e_1, ξ_1, and $\theta \in (0,1)$, and set $e_\theta = \theta e_1 + (1-\theta)e_0$, $\xi_\theta = \theta\xi_1 + (1-\theta)\xi_0$. Our goal is to show that $f(e_\theta, \xi_\theta) \leq \theta f(e_1,\xi_1) + (1-\theta) f(e_0,\xi_0)$. In view of (1), we can choose symmetric, nonnegative matrices M_0 and M_1 such that $f(e_i,\xi_i) = |e_i + \frac{1}{2}\xi_i \otimes \xi_i + M_i|^2$, and our goal can then be written as

$$\min\{|e_\theta + \tfrac{1}{2}\xi_\theta \otimes \xi_\theta + M|^2 \,:\, M \geq 0\} \leq \\ \theta|e_1 + \tfrac{1}{2}\xi_1 \otimes \xi_1 + M_1|^2 + (1-\theta)|e_2 + \tfrac{1}{2}\xi_2 \otimes \xi_2 + M_2|^2. \tag{2}$$

Check that when $M_* = \theta M_1 + (1-\theta)M_0 + \frac{1}{2}\theta(1-\theta)(\xi_1 - \xi_0) \otimes (\xi_1 - \xi_0)$ we have $M_* \geq 0$ and

$$e_\theta + \tfrac{1}{2}\xi_\theta \otimes \xi_\theta + M_* = \theta(e_1 + \tfrac{1}{2}\xi_1 \otimes \xi_1 + M_1) + (1-\theta)(e_2 + \tfrac{1}{2}\xi_2 \otimes \xi_2 + M_2),$$

and show that this implies the validity of (2).

(3) This problem leads you through the most accessible aspects of the problem

$$\min_{\substack{u=0 \text{ at } x=0 \\ u_y = \pm 1}} \int_{0<x<L,\, 0<y<1} u_x^2 + \varepsilon |u_{yy}| \, dx\, dy. \tag{3}$$

Note that this problem is more or less geometric: the domain is divided into two regions, where u_y takes the value $+1$ and -1 respectively, separated by interfaces. The first term prefers that the interfaces be horizontal. The second prefers to avoid interfaces (its value is 2ε times the total length of their projections onto the x axis). The boundary condition at $x = 0$ requires that there be lots of interfaces, at least near $x = 0$. We will show in this problem that the minimum value E_ε is of order $\varepsilon^{2/3} L^{1/3}$.

(a) Consider the behavior of $u(x_0, y)$ as a function of y. Since $u_y = \pm 1$, its graph is a sawtooth, and $\int_0^1 |u_{yy}|\, dy$ is simply two times the number of teeth. Show that if there are N teeth at $x = x_0$, then $\int_0^1 u^2(x_0,y)\, dy \geq cN^{-2}$ (with $c > 0$ independent of N).

(b) Now let $u(x_0, y) = g(y)$, and consider the convex variational problem

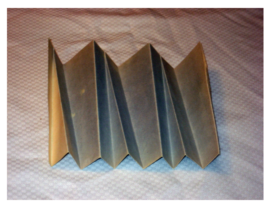
$$\min_{\substack{u=0 \text{ at } x=0 \\ u=g \text{ at } x=x_0}} \int_{0<x<x_0,\, 0<y<1} u_x^2 \, dx\, dy,$$

which is clearly a lower bound for (3). (It plays a role analogous to the relaxed membrane energy discussed in Lecture 2.) Show that the minimizer u_* is affine in x, i.e. that it is $u_*(x,y) = (x/x_0) g(y)$. Conclude that the minimum value is $\frac{1}{x_0^2} \int_0^1 |g(y)|^2\, dy$.

(c) Now consider any u that's admissible for (3), i.e. any $u(x, y)$ such that $u = 0$ at $x = 0$ and $u_y = \pm 1$. If the value of (3) is not too large, then there must clearly be not too many teeth above a typical choice of x_0. But parts (a) and (b) combine to show that if the number of teeth above x_0 is small, then the energy in the region $0 < x < x_0$ is large. Using this tradeoff, show that the minimum of (3) is at least $C\varepsilon^{2/3}L^{1/3}$.

(d) Is this estimate realizable? Provide a heuristic argument that if the spacing of the teeth above x is roughly $\ell(x)$, then $\int u_x^2 \, dx \, dy \sim \int (\ell')^2 \, dx$ and $\int \varepsilon |u_{yy}| \, dx \, dy \sim \int \varepsilon \ell^{-1} \, dx$. (Hint for the former: argue as you did for part (a), to show that for any $x_0 < x_1$, $\int_{x_0 < x < x_1} u_x^2 \, dx \, dy \geq \frac{1}{x_1 - x_0} \int_0^1 |u(x_0, y) - u(x_1, y)|^2 \, dy$). Conclude that the scaling $E_\varepsilon \sim \varepsilon^{2/3} L^{1/3}$ will be achieved when $\ell(x) \sim \varepsilon^{1/3} x^{2/3}$.

(e) Show that the construction suggested by (d) is possible. Hint: the main task is to envision how you can get N teeth at x_1 and $2N$ teeth at $x_0 < x_1$ while preserving the constraint $u_y = \pm 1$ and respecting the heuristic estimates obtained in (d). The answer is a function whose graph looks something like the figure below.



For more on this topic, see R.V. Kohn & S. Müller, Relaxation and regularization of nonconvex variational problems, *Rendiconti del Seminario Matematico e Fisico di Milano* 62 (1992) 89-113 (an expository summary of Kohn and Müller, CPAM 47, 1994, 405-435). Additional references: for the connection between this problem and twinning in martensitic phase transformation see Kohn and Müller, Phil Mag A 66 (1992) 697-715; for the behavior of minimizers near $x = 0$ see S. Conti, CPAM 53 (2000) 1448-1474.