PCMI – Kohn – Problems for TA Session 3

The first two problem sets were long; you may want to use TA Session 3 to return to some of those questions. But here are two more, which are closely related to Lectures 3-5.

(1) In Lectures 3-5 my lower-bound arguments use interpolation inequalities. This problem asks you to prove some of them.

(a) Show that if $u : R \to R$ is periodic with period L, then

$$\int_0^L u_x^2\, dx \leq \left(\int_0^L u^2\, dx\right)^{1/2} \left(\int_0^L u_{xx}^2\, dx\right)^{1/2}.$$

(Hint: this can be done using Fourier series and Plancherel's formula; but it's easier to just use integration by parts).

(b) Dropping the condition of periodicity, show that for any $u : [0, L] \to R$,

$$\int_0^L u_x^2\, dx \leq C_1 \left(\int_0^L u^2\, dx\right)^{1/2} \left(\int_0^L u_{xx}^2\, dx\right)^{1/2} + C_2 L^{-2} \int_0^L u^2\, dx$$

where C_1 and C_2 are constants independent of L. (Hint: this too can be done using integration by parts.)

(c) Show the multidimensional analogue of (a): suppose $u : R^n \to R$ is periodic with period cell $Q = (0, L_1) \times \cdots \times (0, L_n)$, then

$$\int_Q |\nabla u|^2\, dx \leq \left(\int_Q u^2\, dx\right)^{1/2} \left(\int_Q |\nabla\nabla u|^2\, dx\right)^{1/2}.$$

(d) Now suppose $u : R \times [0, L_2] \to R$ is periodic with period L_1 in the first variable (we had a situation like this in Lecture 2, for the out-of-plane displacement of an annulus, viewed as a function of θ and r.) Show that with $Q = (0, L_1) \times (0, L_2)$,

$$\int_Q |\nabla u|^2\, dx \leq C_1 \left(\int_Q u^2\, dx\right)^{1/2} \left(\int_Q |\nabla\nabla u|^2\, dx\right)^{1/2} + C_2 L_2^{-2} \left(\int_Q u^2\, dx\right)$$

where C_1 and C_2 are constants independent of L.

(2) Misfit can make a thin elastic film debond from its substrate, forming a blister. (The misfit could be due, for example, to thermal expansion, if the system is heated and the thermal expansion of the film is much greater than that of the substrate). If D is the blistered region, η is the misfit, and we use a von Karman model with Poisson's ratio zero, then the elastic energy of the blistered region is

$$\min_{w=0, u_3=0, \partial u_3/\partial n=0 \text{ at } \partial D} \int_D |e(w) + \tfrac{1}{2}\nabla u_3 \otimes \nabla u_3 - \eta I|^2 + h^2 |\nabla\nabla u_3|^2\, dx.$$

Ben Belgacem, Conti, DeSimone, and Müller showed that the min is of order $h\eta^{3/2}$ (J Nonlin Sci 10, 2000, 661-685); Jin & Sternberg got the same result around the same time for a closely related problem stated below (J Math Phys 192, 2001, 192-199). This problem asks you to reproduce the lower-bound part of the Jin-Sternberg calculation.

(a) Explain why this problem is essentially the same as the following easier-to-visualize question: *Consider a planar elastic sheet whose stress-free shape is $(1+\eta)D$. Its boundary must be glued to the too-short curve ∂D, uniformly with respect to arclength. Of course the sheet needs to wrinkle a lot near the boundary to accommodate this too-short boundary condition. What is its minimum elastic energy?*

(b) Jin & Sternberg observed that the essential physics doesn't change if we consider a blister that's an infinite strip $0 < x < 1$, and restrict attention to displacements that are periodic in y. Thus, they considered

$$\min_{\substack{w=0, u_3=0, \partial u_3/\partial n=0 \text{ at } x=0,1 \\ w \text{ and } u_3 \text{ periodic in } y}} \int_{(0,1)\times(0,1)} |e(w) + \tfrac{1}{2}\nabla u_3 \otimes \nabla u_3 - \eta I|^2 + h^2 |\nabla\nabla u_3|^2 \, dx. \tag{1}$$

Show that the minimum value of (1) is at least $Ch\eta^{3/2}$, by arguing as follows:

(i) Show that for any $0 < x_0 < 1$,

$$\int_{(0,x_0)\times(0,1)} |\partial_{xy} u_3|^2 \, dx\, dy \geq \frac{1}{x_0} \int_0^1 |\partial_y u_3|^2 (x_0, y)\, dy.$$

(Hint: you did something very similar on problem set 2.)

(ii) Show that if $\int_0^1 \frac{1}{2} |\partial_y u_3|^2 (x,y)\, dy \leq \frac{1}{2}\eta$ for all $x \in (0, x_0)$, then the energy in the region $(0, x_0) \times (0,1)$ is at least $c\eta^2 x_0$.

(iii) Combine (i) and (ii) to show that the energy in the region $0 < x_0 < h/\sqrt{\eta}$ is at least of order $h\eta^{3/2}$.

[Remark: The proof that this lower bound is achievable is too involved to assign as a problem. But the main ideas are a more or less a combination of things you have already seen. As in the "herringbone problem," one can use a combination of shear and periodic one-dimensional wrinkling to achieve $e(w) + \frac{1}{2}\nabla u_3 \otimes \nabla u_3 = \eta I$ pointwise. But for the blister the length scale of the wrinkling must get smaller as one approaches the edge, and change of the length scale costs energy. The successful construction is similar to that of Lecture 3 (self-similar refinement), but the details are different since the blister is not in tension.]