The annulus problem, using a von Karman membrane model

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1 Orientation, and the model

These notes supplement Robert Kohn's Lecture 3 at the 2014 PCMI Graduate Summer School. For background and heuristic arguments, see the pdf of that lecture. For physics-oriented discussions of this topic see [4] and [3]. For upper and lower energy bounds similar to those presented here but using nonlinear elasticity and a rather general membrane model see [1].

We want to study deformations of a thin elastic sheet of annular shape, which is loaded radially at the inner and outer boundary by uniform forces T_{in} and T_{out}, respectively. To keep things simple we consider very simple form of the energy – Föppl-von Kárman energy[1] with zero Poisson ratio. The deformation is then describe by the in-plane displacement $w = (w_1, w_2)$ and out-of-plane displacement u_3, and should minimize

$$E_h(w, u_3) := \int_{\mathcal{A}} \left| e(w) + \frac{1}{2}\nabla u_3 \otimes \nabla u_3 \right|^2 + h^2 \left|\nabla^2 u_3\right| \,\mathrm{d}x + \int_{|x|=R_{in}} T_{in} w(x) \cdot \frac{x}{|x|} \,\mathrm{d}\sigma - \int_{|x|=R_{out}} T_{out} w(x) \cdot \frac{x}{|x|} \,\mathrm{d}\sigma, \quad (1.1)$$

where by R_{in} and R_{out} we denoted the inner and outer radius, respectively; the domain is $\mathcal{A} = \left\{x \in \mathbb{R}^2 : |x| \in (R_{in}, R_{out})\right\}$; $e(w)$ is the linear strain $e(w) = \left(\nabla w + \nabla^t w\right)/2$; and $h > 0$ is the thickness of the sheet.

The domain and applied forces are radially symmetric, and so it will be convenient to use polar coordinates $r \in (R_{in}, R_{out})$ and $\theta \in [0, 2\pi)$. If u_r, u_θ denotes the in-plane part of the displacement in the radial and azimuthal direction, respectively, and ζ denotes the

[1]The FvK viewpoint is appropriate in the small-strain, small-slope regime, and is often used in the physics community.

out-of-plane displacement, the energy has the form

$$E_h(u_r, u_\theta, \zeta) =$$
$$\int_{\mathcal{A}} \begin{pmatrix} \partial_r u_r + \frac{1}{2}(\partial_r\zeta)^2 & \frac{1}{2r}\partial_\theta u_r + \frac{1}{2}\partial_r u_\theta - \frac{1}{2r}u_\theta + \frac{1}{2r}\partial_r\zeta\partial_\theta\zeta \\ \frac{1}{2r}\partial_\theta u_r + \frac{1}{2}\partial_r u_\theta - \frac{1}{2r}u_\theta + \frac{1}{2r}\partial_r\zeta\partial_\theta\zeta & \frac{u_r}{r} + \frac{1}{r}\partial_\theta u_\theta + \frac{1}{2r^2}(\partial_\theta\zeta)^2 \end{pmatrix}^2 r\,\mathrm{d}r\,\mathrm{d}\theta$$
$$+ h^2 \int_{\mathcal{A}} \left(|\partial_{rr}\zeta|^2 + \frac{2}{r^2}|\partial_{r\theta}\zeta|^2 + \frac{1}{r^4}|\partial_{\theta\theta}\zeta|^2 \right) r\,\mathrm{d}r\,\mathrm{d}\theta$$
$$+ T_{in} \int_{r=R_{in}} u_r r\,\mathrm{d}\theta - T_{out} \int_{r=R_{out}} u_r r\,\mathrm{d}\theta. \quad (1.2)$$

The first integrand is called *membrane energy*, the one with prefactor h^2 *bending energy*, and the last two integrals represent applied *loads*.

Let E_{rel} denotes the *relaxation* (Γ-limit) of energy functionals E_h. Then E_{rel} has a unique (up to a translation) minimizer $(\tilde{u}_r, 0, 0)$, i.e. the minimizer is radially symmetric and stays in-plane. In this note we prove the following: Let $\mathcal{E}_0$ denotes the relaxed energy of the minimizer and let the loads $T_{in} > 0$ and $T_{out} > 0$ satisfy (2.3). Then there exist constants $0 < C_0 < C_1$ such that for any $h \in (0, 1)$ we have

$$\mathcal{E}_0 + C_0 h \le \min E_h \le \mathcal{E}_0 + C_1 h. \quad (1.3)$$

Here and also later, all the constant depend only on the data, i.e. radii R_{in}, R_{out} and loads T_{in}, T_{out}. In the following $a \lesssim b$, $a \gtrsim b$, $a \sim b$, will stand for $a \le Cb$, $a \ge Cb$, $C^{-1}a \le b \le Cb$, respectively.

Before we turn to the proof of (1.3), let us write its key steps:

- we write the relaxed energy E_{rel}, and show it has a unique, radially symmetric minimizer $\tilde{u}_r$. Moreover, assumption (2.3) implies that $\tilde{u}_r < 0$ and $\tilde{u}_r > 0$ in connected and non-empty parts of the annulus (we will call these regions *inner* and *outer* region, respectively);
- given any deformation u_r, u_θ, ζ, we compute the difference between its energy $E_h(u_r, u_\theta, \zeta)$ and $\mathcal{E}_0 = E_{rel}(\tilde{u}_r, 0, 0)$. If we denote this *excess energy* by ε, the lower bound in (1.3) is equivalent to showing that $\varepsilon \gtrsim h$;
- we observe that $\int_{\mathcal{A}} |\partial_r\zeta|^2 \lesssim \varepsilon$, which together with control on variations of ζ in the azimuthal direction in part of the annulus implies control on L^2-norm of ζ itself;
- we also observe that $h^2 \int_{\mathcal{A}} |\nabla\nabla\zeta|^2 \lesssim \varepsilon$. Then by interpolation with the previous estimate we get $h \int_{\mathcal{A}} |\nabla\zeta|^2 \lesssim \varepsilon$, in particular $h \int_{\mathcal{A}} |\partial_\theta\zeta|^2 \lesssim \varepsilon$;
- we show that if ε is small (meaning it is smaller than some fixed constant), then u_r is sufficiently close to $\tilde{u}_r$. Since in the inner region $\tilde{u}_r < 0$, the same has to be true for u_r in a slightly smaller region. Then either we have compression in this region (which would contribute $\mathcal{O}(1)$ to the excess energy) or there is a non-trivial out-of-plane displacement ζ in this region – in this case we get that $\int |\partial_\theta\zeta|^2 \gtrsim 1$ in this region, which combined with above implies $\varepsilon \gtrsim h$;

- to prove the upper bound (the second inequality in (1.3)), given $h \in (0,1)$ we need to construct (define) a deformation with energy smaller than $\mathcal{E}_0 + Ch$. A naive first idea is to set $u_r := \tilde{u}_r$ and for ζ use a sinusoidal profile with appropriate radius dependent amplitude and some fixed (optimized) period to release the compression in the azimuthal direction in $\{\tilde{u}_r < 0\}$. By doing this (modulo some cut-off near the transition from wrinkled to planar state) we obtain deformation with energy bounded by $\mathcal{E}_0 + Ch\,(|\log h| + 1)$ – a worse estimate that we expected.
- to get the optimal scaling, we introduce a construction with radius dependent period (i.e. with the radius we change not only the amplitude of wrinkles, but also the wavenumber).

2 The relaxed problem

Using the result of Conti, Maggi, and Müller [2], relaxation of (1.2) can be written as

$$E_{rel}(u_r, u_\theta, \zeta) =$$
$$\int_{\mathcal{A}} \begin{pmatrix} \partial_r u_r + \frac{1}{2}(\partial_r \zeta)^2 & \frac{1}{2r}\partial_\theta u_r + \frac{1}{2}\partial_r u_\theta - \frac{1}{2r}u_\theta + \frac{1}{2r}\partial_r\zeta\partial_\theta\zeta \\ \frac{1}{2r}\partial_\theta u_r + \frac{1}{2}\partial_r u_\theta - \frac{1}{2r}u_\theta + \frac{1}{2r}\partial_r\zeta\partial_\theta\zeta & \frac{u_r}{r} + \frac{1}{r}\partial_\theta u_\theta + \frac{1}{2r^2}(\partial_\theta\zeta)^2 \end{pmatrix}_+^2 + \text{loads}\,. \tag{2.1}$$

Here $(A)_+^2 = \inf\{|A+B|^2 : B = B^t \geq 0\}$ denotes square of the Frobenius norm of the positive part of the matrix. It is easy to see that $(A)_+^2 = (\lambda_1)_+^2 + (\lambda_2)_+^2$, where λ_1,λ_2 are eigenvalues of A, and $f_+ = \max(f, 0)$. This has the following physical interpretation: as $h \to 0$, any compression (which corresponds to the negative part of the strain) can be removed by (infinitesimally) small out-of-plane oscillations (these oscillations "introduce" B). We observe that since the contribution from ζ is a positive matrix, we can assume $\zeta = 0$ while looking for a minimizer of E_{rel}. Moreover, since the problem domain and loads are radially symmetric, it makes sense to first look for a radially symmetric minimizer.

Assuming $\zeta = 0$, $u_\theta = 0$, and $\partial_\theta u_r = 0$, the energy (2.1) simplifies significantly

$$E_{rel}(u_r, 0, 0) = 2\pi \left(\int_{R_{in}}^{R_{out}} r \left((\partial_r u_r)_+^2 + \left(\frac{u_r}{r}\right)_+^2 \right) \mathrm{d}r + T_{in} R_{in} u_r(R_{in}) - T_{out} R_{out} u_r(R_{out}) \right). \tag{2.2}$$

We want to find conditions on the data (radii and forces) so that u_r, minimizer of (2.2), is negative in part of the domain. We will show the following:

Lemma. *Let $T_{in} > 0$ and $T_{out} > 0$ satisfy*

$$R_{in} T_{in} < R_{out} T_{out} \quad \textit{and} \quad \frac{T_{in}}{T_{out}} > 2\frac{R_{out}^2}{R_{in}^2 + R_{out}^2}. \tag{2.3}$$

Then (2.2) has a unique minimizer $\tilde{u}_r$. Moreover, there exists $L \in (R_{in}, R_{out})$ such that

$$\tilde{u}_r < 0 \textit{ in } (R_{in}, L), \qquad \tilde{u}_r > 0 \textit{ in } (L, R_{out}). \tag{2.4}$$

We also have

$$\partial_r \tilde{u}_r \geq \frac{R_{in}}{R_{out}} T_{in} \quad \textit{and} \quad \partial_r \left(r \partial_r \tilde{u}_r(r) \right) = \frac{\tilde{u}_r(r)_+}{r} \tag{2.5}$$

and for $r \in (R_{in}, L)$ *the function* $\beta(r) := (-\tilde{u}_r(r) r)^{\frac{1}{2}}$ *satisfies*

$$|\beta(r)| \lesssim (L-r)^{\frac{1}{2}}, \quad |\partial_r \beta(r)| \lesssim (L-r)^{-\frac{1}{2}}, \quad |\partial_{rr}\beta(r)| \lesssim (L-r)^{-\frac{3}{2}}. \tag{2.6}$$

Proof. The proof is elementary (but lengthy), and will be given in the last section. □

To observe that u_r is a minimizer of (2.1), we use that $(u_r, 0, 0)$ is a critical point of (2.1) and that (2.1) is a convex functional (see [2]). It will be clear later that in fact it is the unique minimizer (up to a translation), i.e. the only solution to the relaxed problem is radially symmetric.[2] Then

$$\begin{aligned}
&E_h(u_r, u_\theta, \zeta) - E_{rel}(\tilde{u}_r, 0, 0) = E_h(u_r, u_\theta, \zeta) - \left(\int_{\mathcal{A}} \begin{pmatrix} \partial_r \tilde{u}_r & 0 \\ 0 & \frac{(\tilde{u}_r)_+}{r} \end{pmatrix}^2 + \text{ loads} \right) \\
&= \int_{\mathcal{A}} \partial_r \tilde{u}_r \left(\partial_r \zeta\right)^2 + \left(\partial_r \tilde{u}_r - \partial_r u_r - \frac{1}{2} \left(\partial_r \zeta\right)^2 \right)^2 \\
&\quad + \frac{1}{r^2} \left(\frac{(\tilde{u}_r)_+}{r} \right) \left(\partial_\theta \zeta\right)^2 + \left(\frac{(\tilde{u}_r)_+}{r} - \left(\frac{u_r}{r} + \frac{1}{r} \partial_\theta u_\theta + \frac{1}{2r^2} \left(\partial_\theta \zeta\right)^2 \right) \right)^2 \\
&\quad + 2 \left(\frac{1}{2r} \partial_\theta u_r + \frac{1}{2} \partial_r u_\theta - \frac{1}{2r} u_\theta + \frac{1}{2r} \partial_r \zeta \partial_\theta \zeta \right)^2 + h^2 \left(|\partial_{rr}\zeta|^2 + \frac{2}{r^2} |\partial_{r\theta}\zeta|^2 + \frac{1}{r^4} |\partial_{\theta\theta}\zeta|^2 \right) \\
&=: \varepsilon \geq 0,
\end{aligned} \tag{2.7}$$

where we used (2.5). Using Taylor expansion of E_h around the relaxed solution, the above relation can be understood as follows: since $\tilde{u}_r$ is the solution to the relaxed problem (in particular critical point), the first variation of E_h wrt the in-plane directions (i.e. in u_r and u_θ) vanishes, and only the quadratic terms (which in our setting are very simple since the second variation in the strain is just twice the identity) remain. We know that the contribution from the out-of-plane displacement ζ can only increase the energy, which is consistent with the above equality (recall that $\partial_r \tilde{u}_r > 0$ and $(\tilde{u}_r)_+ \geq 0$).

3 Lower bound

Our goal is to prove a lower bound $\varepsilon \gtrsim \min(1, h)$ (which implies $\varepsilon \gtrsim h$ since we consider only $h \leq 1$).

By (2.5) $\partial_r \tilde{u}_r$ is strictly larger than 0, and so (2.7) implies

$$\int_{\mathcal{A}} (\partial_r \zeta)^2 \lesssim \varepsilon. \tag{3.1}$$

[2] From the physical point of view this is natural, since we expect the symmetry will break only by wrinkles – i.e. if $h > 0$.

Since $\tilde{u}_r > 0$ in (L, R_{out}), we have $\tilde{u}_r \geq \delta > 0$ in $\mathcal{O} = ((L + R_{out})/2, R_{out}) \times [0, 2\pi)$, and (2.7) implies

$$\int_{\mathcal{O}} (\partial_\theta \zeta)^2 \lesssim \int_{\mathcal{O}} \frac{1}{r^2} \left(\frac{\tilde{u}_r}{r} \right) (\partial_\theta \zeta)^2 \leq \varepsilon. \tag{3.2}$$

We combine two previous estimates to get $\int_{\mathcal{O}} |\nabla \zeta|^2 \lesssim \varepsilon$. Application of Poincaré inequality gives $\int_{\mathcal{O}} |\zeta - \bar{\zeta}|^2 \lesssim \varepsilon$, which using (3.1) can be upgraded to the whole $\mathcal{A}$:

$$\int_{\mathcal{A}} |\zeta - \bar{\zeta}|^2 \lesssim \varepsilon. \tag{3.3}$$

From (2.7) we get that $\int_{\mathcal{A}} |\nabla \nabla \zeta|^2 \leq \varepsilon h^{-2}$, and so by interpolation between that and (3.3) we get $\int_{\mathcal{A}} |\nabla \zeta|^2 \lesssim \varepsilon h^{-1}$, and in particular

$$\int_{\mathcal{A}} (\partial_\theta \zeta)^2 \lesssim \varepsilon h^{-1}. \tag{3.4}$$

Now we want to show that $\int_{\mathcal{A}} (\partial_\theta \zeta)^2 \gtrsim 1$ provided $\varepsilon \lesssim 1$, which in turn will imply the desired bound on ε. Let $v := \tilde{u}_r - u_r$. We want to show that

$$F(r) := \int_0^{2\pi} v(r, \theta) \, \mathrm{d}\theta \tag{3.5}$$

is small for all $r \in (R_{in}, R_{out})$. By (2.7)

$$\int_{\mathcal{O}} \left(\frac{v}{r} - \frac{1}{r} \partial_\theta u_\theta - \frac{1}{2r^2} (\partial_\theta \zeta)^2 \right)^2 \leq \varepsilon, \tag{3.6}$$

and so

$$\left| \int_{(L+R_{out})/2}^{R_{out}} \frac{F(r)}{r} \, \mathrm{d}r \right| = \left| \int_{\mathcal{O}} \frac{v}{r} \right| \leq \left| \int_{\mathcal{O}} \frac{v}{r} - \frac{1}{r} \partial_\theta u_\theta - \frac{1}{2r^2} (\partial_\theta \zeta)^2 \right| + \frac{1}{2r^2} \left| \int_{\mathcal{O}} (\partial_\theta \zeta)^2 \right| \overset{(3.6),(3.2)}{\lesssim} \varepsilon^{1/2} + \varepsilon. \tag{3.7}$$

By (2.7) we have $\int_{\mathcal{A}} \left(\partial_r v - \frac{1}{2} (\partial_r \zeta)^2 \right)^2 \leq \varepsilon$, which together with (3.1) and using the same idea as above gives $|F(r_0) - F(r_1)| \lesssim \varepsilon + \varepsilon^{1/2}$. Therefore, by (3.7) we get for any $r \in (R_{in}, R_{out})$

$$\left| \int_0^{2\pi} \tilde{u}_r(r) - u_r(r, \theta) \, \mathrm{d}\theta \right| = |F(r)| \lesssim \varepsilon + \varepsilon^{1/2}. \tag{3.8}$$

Now let $\mathcal{I} := (R_{in}, (R_{in} + L)/2) \times [0, 2\pi)$. We write

$$\int_{\mathcal{I}} \frac{1}{2r^2} (\partial_\theta \zeta)^2 = \int_{\mathcal{I}} \left(\frac{u_r}{r} + \frac{1}{r} \partial_\theta u_\theta + \frac{1}{2r^2} (\partial_\theta \zeta)^2 \right) + \int_{\mathcal{I}} \left(\frac{\tilde{u}_r}{r} - \frac{u_r}{r} \right) - \int_{\mathcal{I}} \frac{\tilde{u}_r}{r}. \tag{3.9}$$

Since $\tilde{u}_r < 0$ in $\mathcal{I}$, by (2.7) $\int_{\mathcal{I}} \left(\frac{u_r}{r} + \frac{1}{r} \partial_\theta u_\theta + \frac{1}{2r^2} (\partial_\theta \zeta)^2 \right)^2 \leq \varepsilon$, and so

$$\left| \int_{\mathcal{I}} \frac{1}{2r^2} (\partial_\theta \zeta)^2 + \int_{\mathcal{I}} \frac{\tilde{u}_r}{r} \right| \leq \int_{\mathcal{I}} \left| \frac{u_r}{r} + \frac{1}{r} \partial_\theta u_\theta + \frac{1}{2r^2} (\partial_\theta \zeta)^2 \right| + \left| \int_{\mathcal{I}} \left(\frac{\tilde{u}_r}{r} - \frac{u_r}{r} \right) \right| \lesssim \varepsilon^{1/2} + \varepsilon. \tag{3.10}$$

Moreover, $-\tilde{u}_r \geq \kappa > 0$ in $\mathcal{I}$, which gives

$$\int_{\mathcal{I}} (\partial_\theta \zeta)^2 \gtrsim \kappa - C(\varepsilon + \varepsilon^{1/2}), \tag{3.11}$$

where C and κ depend only on the data. Combining this with (3.4) gives

$$\varepsilon h^{-1} \gtrsim \kappa - C(\varepsilon + \varepsilon^{1/2}), \tag{3.12}$$

and so either $\varepsilon + \varepsilon^{1/2} \geq \kappa/2C$ (i.e. $\varepsilon \gtrsim 1$) or $\varepsilon h^{-1} \geq \kappa/2$. We showed that $\varepsilon \gtrsim \min(1, h)$, which completes the proof of the lower bound.

4 Upper bound

To show the upper bound, for any $h > 0$ we will construct a deformation with excess energy bounded by Ch. Since we are not trying to get optimal constant, it is enough to the construction only for $h = k^{-2}$, $k \in \mathbb{N}$. Moreover, w.l.o.g. we can assume $h \leq L - R_{in}$.

We first define ζ and obtain estimates for some of its derivatives. We use ζ to define u_θ, and use $u_r := \tilde{u}_r$. Finally, we estimate the excess energy ε.

Step 1: Let $\varphi : [0, \infty) \to [0, 1]$ be a smooth function with support in $(1/4, 4)$ such that for all $x \in (0, 1]$

$$\sum_{n=0}^{\infty} \varphi^2 \left(4^n x\right) = 1. \tag{4.1}$$

Let $N \in \mathbb{N}$ be chosen such that $\frac{h}{4} \leq 4^{-N}(L - R_{in}) \leq h$. Then we define

$$\zeta\left(r, \theta\right) := 2\sqrt{2\pi} h^{\frac{1}{2}} \left((-\tilde{u}_r(r))r\right)^{\frac{1}{2}} \sum_{n=0}^{N} \varphi\left(\frac{L - r}{L - R_{in}} 4^n\right) 2^{-n} \cos\left(2^n h^{-\frac{1}{2}} \theta\right) \tag{4.2}$$

for $r \in (R_{in}, L)$ and $\zeta = 0$ otherwise.

Step 2: Formal estimates on ζ and the excess energy ε.

Before we do rigorous but tedious estimates of ζ let us present some heuristics. Looking at (4.2) and using that φ is supported in $(1/4, 4)$, we see that for any $r \in (R_{in}, L)$ at most two terms in the sum are present. Hence, with a little simplification we can think of ζ as a sinusoidal curve with radius-dependent amplitude and period. Accepting this viewpoint, the amplitude and period are easy to compute:

Since $\varphi\left(\frac{L-r}{L-R_{in}} 4^n\right) \neq 0$ only if $\frac{L-r}{L-R_{in}} 4^n \in (1/4, 4)$, we see that $|\zeta| \sim h^{1/2} \left(-\tilde{u}_r(r) r\right)^{\frac{1}{2}} 2^{-n} \sim h^{1/2}(L - r)$, where we used that $(-\tilde{u}_r(r)r)^{\frac{1}{2}} = \beta(r) \sim (L - r)^{\frac{1}{2}}$ and $2^{-n} \sim (L - r)^{\frac{1}{2}}$. The period is simply $2^{-n} h^{1/2} \sim h^{1/2} \left(L - r\right)^{\frac{1}{2}}$.

Having this, it is straightforward to guess what estimates on ζ one could expect. We already have

$$\zeta(r, \theta) \sim h^{\frac{1}{2}} (L - r). \tag{4.3}$$

Taking formally derivatives in r gives

$$\partial_r \zeta(r, \theta) \sim h^{\frac{1}{2}} \quad \text{and} \quad \partial_{rr} \zeta(r, \theta) \sim h^{\frac{1}{2}} (L - r)^{-1}. \tag{4.4}$$

Since the period is of order $h^{\frac{1}{2}}(L-r)^{\frac{1}{2}}$, taking derivative in θ is as dividing by the period, and so the above suggests

$$\partial_\theta \zeta(r,\theta) \sim (L-r)^{\frac{1}{2}}, \qquad \partial_{r\theta}\zeta(r,\theta) \sim (L-r)^{-\frac{1}{2}}, \qquad \text{and} \qquad \partial_{\theta\theta}\zeta(r,\theta) \sim h^{-\frac{1}{2}}. \tag{4.5}$$

Moreover, ζ is defined in such a way that

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2r^2}(\partial_\theta \zeta)^2 \, \mathrm{d}\theta = \frac{-\tilde{u}_r}{r} \tag{4.6}$$

for $r \in (R_{in}, L - 4^{-N}(L-R_{in}))$.

Before we prove the above estimates (at least the upper bounds), let us show how these estimates imply the upper bound on the excess energy ε. To do that we will first define u_θ (and take $u_r = \tilde{u}_r$) and estimate all the terms in (2.7).

In order to have $\frac{u_r}{r} + \frac{1}{r}\partial_\theta u_\theta + \frac{1}{2r^2}(\partial_\theta\zeta)^2 = 0$ (at least for $r \in (R_{in}, L - 4^N(L-R_{in}))$), for $r \in (R_{in}, L)$ we define

$$u_\theta(r,\theta) := -\frac{1}{2r}\int_0^\theta \left((\partial_\theta\zeta)^2 - \left(\frac{1}{2\pi}\int_0^{2\pi} (\partial_\theta\zeta)^2 \right) \right) \mathrm{d}\theta. \tag{4.7}$$

Since ζ is $h^{\frac{1}{2}}$-periodic in θ, by definition u_θ is also $h^{\frac{1}{2}}$-periodic. That means in the following estimates we can assume $|\theta| \le h^{\frac{1}{2}}$. By (4.5) we have

$$|u_\theta(r,\theta)| \lesssim |\theta|\,(L-r) \le h^{\frac{1}{2}}(L-r). \tag{4.8}$$

Similarly, for $\partial_r u_\theta$ we get

$$|\partial_r u_\theta(r,\theta)| \lesssim |\theta| \left(\|\partial_\theta\zeta(r,\cdot)\|_{L^\infty}^2 + \|\partial_\theta\zeta(r,\cdot)\|_{L^\infty}\|\partial_{r\theta}\zeta(r,\cdot)\|_{L^\infty} \right) \lesssim h^{\frac{1}{2}}. \tag{4.9}$$

Now we can estimate terms in (2.7):

- $\partial_r\tilde{u}_r \lesssim 1$, and so by (4.4) we have $\int_{\mathcal{A}} \partial_r\tilde{u}_r\,(\partial_r\zeta)^2 \lesssim h$;
- $\tilde{u}_r = u_r$, and so by (4.4) we have $\int_{\mathcal{A}} \left(\partial_r\tilde{u}_r - \partial_r u_r - \frac{1}{2}(\partial_r\zeta)^2 \right)^2 = \frac{1}{4}\int_{\mathcal{A}} (\partial_r\zeta)^4 \lesssim h^2$;
- $(u_r)_+$ and ζ have distinct support, and so $\frac{1}{r^2}\left(\frac{(\tilde{u}_r)_+}{r}\right)(\partial_\theta\zeta)^2 = 0$ everywhere;
- the term $\left(\frac{(\tilde{u}_r)_+}{r} - \left(\frac{u_r}{r} + \frac{1}{r}\partial_\theta u_\theta + \frac{1}{2r^2}(\partial_\theta\zeta)^2 \right) \right)^2$ vanishes except for a small region $r \in (L - 4^{-N}(L-R_{in}), L)$ (a cut-off region, with size of order h), where it is enough to estimate the term pointwise by a constant (but this term is, in fact, much smaller – of order h^2);
- by (4.8),(4.9),(4.4), and (4.5) we have $\left(\frac{1}{2r}\partial_\theta u_r + \frac{1}{2}\partial_r u_\theta - \frac{1}{2r}u_\theta + \frac{1}{2r}\partial_r\zeta\partial_\theta\zeta \right)^2 \lesssim h$;
- finally, by (4.4) and (4.5) the bending term is bounded by a multiple of $h^2\int_{h/4}^{L-R_{in}} \frac{h}{r^2} + r^{-1} + h^{-1}\,\mathrm{d}r \lesssim h$.

This concludes the proof of the upper bound provided we rigorously show (4.3), (4.4), and (4.5).

Step 3: Proof of (4.3-4.5). As long as $L-r \geq (L-R_{in})4^{-N}$ we have $\sum_{n=0}^{N}\varphi^2\left(\frac{L-r}{L-R_{in}}4^n\right) = 1$, and so for such r

$$\begin{aligned}\int_0^{2\pi}(\partial_\theta\zeta)^2\,\mathrm{d}\theta &= 4\,(2\pi)\,h\,((-\tilde{u}_r(r))r)\sum_{n=0}^{N}\varphi^2\left(\frac{L-r}{L-R_{in}}4^n\right)2^{-2n}\int_0^{2\pi}\sin^2\left(2^n h^{-\frac{1}{2}}\theta\right)2^{2n}h^{-1}\,\mathrm{d}\theta\\ &= 4\pi(-\tilde{u}_r(r)r), \end{aligned}\tag{4.10}$$

where we used that $h^{-\frac{1}{2}}$ is an integer and that $\sin(k\theta)$ are orthogonal in $L^2(0,2\pi)$. Moreover, it follows from (4.2) that

$$\zeta(r,\cdot)\text{ is }h^{1/2}\text{-periodic.}\tag{4.11}$$

Next

$$|\partial_r\zeta(r,\theta)| \lesssim h^{1/2}\sum_{n=0}^{N}\partial_r\left(\beta(r)\varphi\left(\frac{L-r}{L-R_{in}}4^n\right)\right)2^{-n},\tag{4.12}$$

where $\beta(r)$ was define in (2.6). Since φ has support in $(1/4,4)$, we see that $\varphi\left(\frac{L-r}{L-R_{in}}4^n\right)\neq 0$ only if $4^{-n}\sim L-r$. Moreover, at most 2 terms in the sum are not zero, and so $\left|\partial_r\left(\beta(r)\varphi\left(\frac{L-r}{L-R_{in}}4^n\right)\right)\right|2^{-n} \overset{(2.6)}{\lesssim} \left((L-r)^{-\frac{1}{2}}+(L-r)^{\frac{1}{2}}4^n\right)2^{-n}\lesssim 1$ implies

$$|\partial_r\zeta(r,\theta)|\lesssim h^{1/2}.\tag{4.13}$$

For $\partial_\theta\zeta$ we directly have

$$|\partial_\theta\zeta(r,\theta)|\lesssim (L-r)^{\frac{1}{2}}\,,\tag{4.14}$$

where we again used that at most two terms in the sum in (4.2) are not zero. For the second derivatives of ζ we similarly obtain

$$|\partial_{\theta\theta}\zeta|\lesssim h^{\frac{1}{2}}\beta(r)\sum_{n=0}^{N}\varphi\left(\frac{L-r}{L-R_{in}}4^n\right)2^{-n}2^{2n}h^{-1}\lesssim h^{-\frac{1}{2}},\tag{4.15}$$

where we again used that only the terms for which $(L-r)\sim 4^{-n}$ are present, and so β, which by (2.6) is smaller than $C(L-r)^{\frac{1}{2}}$, cancelled with $\left(2^{-n}2^{2n}\right)$. Next

$$|\partial_{r\theta}\zeta|\lesssim\left|\partial_r\left(\beta(r)\varphi\left(\frac{L-r}{L-R_{in}}4^n\right)\right)\right|\lesssim 2^n\lesssim (L-r)^{-\frac{1}{2}}\,,\tag{4.16}$$

where as before n is such that $4^{-n}\sim L-r$. Finally we have

$$|\partial_{rr}\zeta(r,\theta)|\lesssim h^{\frac{1}{2}}2^{-n}\left|\partial_{rr}\left(\sqrt{-u_r r}\,\varphi\left(\frac{L-r}{L-R_{in}}4^n\right)\right)\right|\lesssim\frac{h^{\frac{1}{2}}}{L-r},\tag{4.17}$$

where again n is such that $4^{-n}\sim L-r$.

5 Proof of the Lemma

First we observe that if $R_{in}T_{in} > R_{out}T_{out}$ the energy (2.2) is not bounded by below, so in the following we assume $R_{in}T_{in} \geq R_{out}T_{out}$. Let $f(r) := \frac{R_{out}-r}{R_{out}-R_{in}} R_{in}T_{in} + \frac{r-R_{in}}{R_{out}-R_{in}} R_{out}T_{out}$ be a linear function interpolating between $T_{in}R_{in}$ at $r = R_{in}$ and $T_{out}R_{out}$ at $r = R_{out}$. Then clearly $f > 0$ and $\partial_r f \geq 0$ in (R_{in}, R_{out}). Since $\partial_r(f(r)u_r(r)) = r\left(\partial_r f \frac{u_r}{r} + \frac{f}{r}\partial_r u_r\right)$, the energy (2.2) can be written as

$$E_{rel}(u_r, 0, 0)/2\pi = \int_{R_{in}}^{R_{out}} r\left((\partial_r u_r)_+^2 - \frac{f}{r}\partial_r u_r + \left(\frac{u_r}{r}\right)_+^2 - \partial_r f\frac{u_r}{r}\right) \mathrm{d}r. \tag{5.1}$$

Since both f and $\partial_r f$ have the correct sign, we see that the energy is bounded from below and that any minimizing sequence $\partial_r u_r$ is bounded in $L^2(R_{in}, R_{out})$. We see that if $T_{in}R_{in} = T_{out}R_{out}$, $\partial_r f = 0$ and the variational problem is degenerate and possesses infinitely many solutions of the form $u_r(r) := \alpha$ for all $\alpha \leq 0$.

Let us therefore consider the case $T_{in}R_{in} < T_{out}R_{out}$. In this case $f > 0$ and $\partial_r f > 0$, and the minimizing sequence is bounded in $W^{1,2} \cap L^\infty(R_{in}, R_{out})$. Hence the existence of a minimizer can be shown using direct method of Calculus of Variations. We observe that the minimizer has to satisfy $\partial_r u_r \geq 0$ a.e. Indeed, it this were not the case, we define $v(r) := u_r(R_{out}) - \int_r^{R_{out}} (\partial_r u_r(r'))_+ \,\mathrm{d}r'$ so that $(\partial_r v)_+ = (\partial_r u_r)_+$, $v_+ \leq (u_r)_+$. Since $v(R_{in}) < u_r(R_{in})$, energy of v would be strictly smaller than of u_r.

Now we replace the first integrand in the energy functional by $(\partial_r u_r)^2$. Since $\partial_r u_r \geq 0$ a.e., we see that u_r will still be minimizer of this new (possibly larger) energy functional. The advantage of this functional over the previous is that we can integrate by parts to derive the Euler-Lagrange equations:

$$\partial_r\left(r\partial_r u_r(r)\right) = \frac{u_r(r)_+}{r}, \tag{5.2}$$

with the boundary conditions $2\partial_r u_r(R_{in}) = T_{in}$ and $2\partial_r u_r(R_{out}) = T_{out}$. Since $(u_r)_+ \geq 0$, we see that $r\partial_r u_r(r)$ is non-decreasing and (2.5) follows. Hence u_r is strictly increasing, and so $u_r < 0$ in (R_{in}, L) and $u_r > 0$ in (L, R_{out}) (where possibly $L = R_{in}$ or $L = R_{out}$).

We would like to derive conditions on the data so that $R_{in} < L < R_{out}$. In the case $L = R_{out}$ the quantity $ru_r(r)$ would be constant, in particular we would have $T_{in}R_{int} = T_{out}R_{out}$. Hence it remains to rule out the case $L = R_{in}$ (no azimuthal compression). Using the Euler-Lagrange equation and the boundary values we compute explicitly the solution

$$2u_r(r) = \frac{T_{in} - T_{out}}{R_{out}^{-2} - R_{in}^{-2}}\frac{1}{r} + \frac{T_{out}R_{out}^2 - T_{in}R_{in}^2}{R_{out}^2 - R_{in}^2}r, \tag{5.3}$$

and observe that $u_r(R_{in}) < 0$ provided the second condition in (2.3) holds.

Since for $r \in (R_{in}, L)$, $u_r(r) = C\log\left(\frac{r}{L}\right)$ for with some $C > 0$, (2.6) follows by a direct computation.

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