PDE for Finance – Homework 2, distributed 2/10/99, due 2/24/99.

1) Let Ω be the square $(-1,1)\times(-1,1)$ in R^2, and for $x\in\Omega$ consider $u(x)=\text{dist}(x,\partial\Omega)$. Show, directly from the definition, that u is a viscosity solution of $|\nabla u|=1$ in the sense that

$$\begin{aligned} u-\phi \text{ has a local max} &\Rightarrow |\nabla\phi|-1\le 0\\ u-\phi \text{ has a local min} &\Rightarrow |\nabla\phi|-1\ge 0. \end{aligned}$$

(We use here the usual shorthand: ϕ is smooth, the local min or max is achieved at some point x_0, and the condition on $\nabla\phi$ is to hold at the same x_0.)

2) When defining the viscosity solution of a time-independent Hamilton-Jacobi equation such as $|\nabla u|=1$, it is natural to wonder which way the inequalities should go. Answer this by relating the value function of a minimum-time problem to that of a finite horizon problem, as follows. Let $u(x)$ be value function of the minimum-time problem with target E, state equation $dy/ds=f(y,\alpha)$, and admissible controls $\alpha(s)\in A$. Let $v(x,t)$ be the value function of the finite horizon problem with the same state equation, the same set of admissible controls, and objective

$$\min_\alpha \int_t^T h_E(y(s))\,ds$$

where

$$h_E(x)=\begin{cases} 1 & \text{if } x\notin E\\ 0 & \text{if } x\in E.\end{cases}$$

Show that v can be expressed in terms of u. Use this to explain the proper definition of a viscosity solution for the Hamilton-Jacobi equation satisfied by u.

3) Consider the solution of

$$\begin{aligned} -\epsilon u_{xx}+u_x^2 &= 1 \quad \text{for } -1<x<1\\ u &= 0 \quad \text{at } x=\pm 1. \end{aligned}$$

with $\epsilon>0$. It is known (and you need not prove) that u is smooth.

(a) Show that $u\ge 0$. (Hint: what properties do u_x and u_{xx} have at the point where u achieves its minimum?)

(b) Show that the solution is unique. (Hint: a special case of the *maximum principle* says: if w satisfies $w_{xx}+g(x)w_x=0$ for $-1<x<1$ then w achieves its maximum and minimum values at the endpoints $x=\pm 1$. You may use this fact without proving it. We'll discuss such things later in the semester; but the proof in this special case is easy – it requires just one new idea beyond the one you used in (a) – and it can be found in any standard PDE book, for example Evans or John. I encourage you to figure out the proof and/or read up on it.)

(c) Show that $u(x)=u(-x)$, and that u_x vanishes only at $x=0$. Conclude that $|u_x|\le 1$.

(d) Solve for u explicitly, using that $v=u_x$ satisfies $-\epsilon v_x+v^2=1$ for $-1<x<0$ and $v=0$ at $x=0$.

(e) The solution considered above depends on ϵ, so let's call it u_ϵ. Show that as $\epsilon\to 0$, u_ϵ tends to $1-|x|$.