PDE for Finance – Homework 4, distributed 3/24/99, due 4/7/99.

1) [Oksendal, Problem 4.5]. Consider

$$\beta_k(t) = E[w^k(t)]$$

where $w(t)$ is Brownian motion (with $w(0) = 0$). Show using Ito's formula that for $k = 2, 3, \ldots$,

$$\beta_k(t) = \frac{1}{2}k(k-1)\int_0^t \beta_{k-2}(s)\, ds.$$

Deduce that $E[w^4(t)] = 3t^2$. What is $E[w^6(t)]$? (The moments of w can also be calculated from its distribution function, since $w(t)$ is Gaussian with mean 0 and variance 1. But the method in this problem is easier, and good practice with Ito's lemma.)

2) [Oksendal, Problem 4.13]. Let y solve

$$dy = f(t)dt + dw$$

and define

$$z(t) = \exp\left(-\int_0^t f\, dw - \frac{1}{2}\int_0^t f^2(r)\, dr\right).$$

Show that $E[y(t)z(t)]$ is independent of time. (This is a special case of Girsanov's theorem.)

3) [Oksendal Problem 5.6] Solve the scalar stochastic differential equation

$$dy = \mu dt + \sigma y dw$$

when μ and σ are constant. [Hint: multiply the equation by the "integrating factor" $\exp(-\sigma w + \frac{1}{2}\sigma^2 t)$.]

4) Consider the mean-reverting Ornstein-Uhlenbeck process

$$dr = a(b - r)dt + \sigma dw$$

where a, b, and σ are constants. Find the mean $E[r(t)]$ and the variance $E[(r - E[r(t)])^2]$.

5) [A lot like Oksendal Problem 7.1]. Find the backward Kolmogorov equation associated with each process:

(a) The Ornstein-Uhlenbeck process $dy = \mu y dt + \sigma dw$.

(b) The Cox-Ingersoll-Ross interest rate model $dr = a(b - r)dt + \sigma\sqrt{r}dw$.

(c) The vector-valued process $dy_i = r_i y_i dt + y_i \sum_j \alpha_{ij} dw_j$.

[Comment on (b): the attraction of this model is that r stays positive, if $ab > \sigma^2/2$; Exercise 34 of Lamberton & Lapeyre sketches a proof.]

6) [Approximately Oksendal's Problem 7.2]. Find the stochastic differential equation for which each is the backward Kolmogorov equation:

(a) $u_t + u_x + u_{xx} = 0$

(b) $u_t + \mu x u_x + \frac{1}{2}\alpha^2 x^2 u_{xx} = 0$

(c) $u_t + 2x_2 \frac{\partial u}{\partial x_1} + \log(1+|x|^2)\frac{\partial u}{\partial x_2} + \frac{1}{2}(1+x_1^2)\frac{\partial^2 u}{\partial x_1^2} + x_1 \frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{1}{2}\frac{\partial^2 u}{\partial x_2^2} = 0.$

7) Let D be a region in R^n, and consider

$$
\begin{aligned}
u(x,t) \quad = \quad & \text{Probability that a Brownian particle starting from } x \in D \\
& \text{at time } t \text{ reaches } \partial D \text{ before time } T.
\end{aligned}
$$

What partial differential equation does u solve? (Be sure to specify the final-time and boundary conditions.)