PDE for Finance – Homework 5, distributed 4/14/99, due 4/28/99.
NO EXTENSIONS

1) Consider the linear heat equation $f_t - f_{xx} = 0$ in one space dimension, with "Heaviside function" initial data

$$f(x,0) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Show by evaluating the solution formula that

$$f(x,t) = \frac{1}{2}\left[1 + \phi(x/\sqrt{4t})\right]$$

where ϕ is the "error function"

$$\phi(s) = \frac{2}{\sqrt{\pi}} \int_0^s e^{-r^2}\, dr.$$

2) Consider the linear heat equation $f_t - \Delta f = 0$ in R^n, with continuous initial data $f_0(x)$ at $t = 0$.

(a) Show that if f_0 is uniformly bounded ($|f_0(x)| \le M$ for all $x \in R^n$) then $f(x,t) \to 0$ as $t \to \infty$.

(b) Show, by giving a counterexample, that if f_0 is not uniformly bounded the conclusion of (a) can be false.

3) Consider the Black-Scholes PDE

$$u_t + \tfrac{1}{2}\sigma^2 s^2 u_{ss} + rsu_s - ru = 0$$

with final value

$$u(s,T) = \Phi(s).$$

If you know some finance, you know that $u(s,t)$ gives the time-t value of a European option with maturity T and payoff Φ, when the time-t value of the underlying asset is s. Here r is the risk-free rate and σ is the volatility of the underlying asset, both assumed constant.

(a) Show using the Feynman-Kac formula that this is the PDE describing

$$u(s,t) = E_{y(t)=s}\left[e^{-r(T-t)}\Phi(y(T))\right]$$

where y solves the stochastic differential equation

$$dy = rydt + \sigma ydw.$$

(b) We know (using Ito's lemma) that $y(t) = e^{z(t)}$ where

$$dz = (r - \tfrac{1}{2}\sigma^2)dt + \sigma dw.$$

Deduce using the Feynman-Kac formula that the function $v(x,t)$ defined by $v(x,t) = u(e^x, t)$ solves the constant-coefficient PDE

$$v_t + \tfrac{1}{2}\sigma^2 v_{xx} + (r - \tfrac{1}{2}\sigma^2)v_x - rv = 0$$

with final value $v(x,T) = \Phi(e^x)$.

(c) Show there exist constants α and β such that $w(x,t) = e^{\alpha x + \beta t} v(x,t)$ solves the backward-in-time linear heat equation

$$w_t + \tfrac{1}{2}\sigma^2 w_{xx} = 0.$$

4) Consider the linear heat equation $f_t - f_{xx} = 0$ on the half-line $x > 0$, with boundary condition $f(0,t) = 0$. Assume the initial data $f_0(x) = f(x,t)$ satisfies $f_0(0) = 0$. Show the solution is

$$f(x,t) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty f_0(z) \left(e^{-(x-z)^2/(4t)} - e^{-(x+z)^2/(4t)} \right) dz.$$

(Hint: consider the initial value problem on R, with the odd extension of f_0 as initial data; show its solution vanishes at $x = 0$ for all $t > 0$.) [Comment: this solution formula, together with a reduction like that of Problem 3, leads to explicit values for European-style barrier options.]

5) Consider once again the linear heat equation $f_t - f_{xx} = 0$ on the half-line $x > 0$. This time we take the initial data to be zero ($f(x,0) = 0$) and we specify nonzero boundary data $f(0,t) = g(t)$ with $g(0) = 0$. Show that

$$f(x,t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-r)^{3/2}} e^{\frac{-x^2}{4(t-r)}} g(r) dr.$$

(Hint: consider the function $h(x,t) = f(x,t) - g(t)$, extended to $x < 0$ by odd reflection.)

6) Consider the initial-boundary-value problem for

$$u_t = a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u$$

with x in the interval $D = (0,1)$ and $0 < t < T$. We assume that u, a, b, and c are sufficiently smooth, and that $a(x,t) > 0$.

(a) Show that if $c < 0$ for all x and t then $|u|$ achieves its maximum at the "initial boundary" $t = 0$ or at the "spatial boundary" $x = 0, 1$. (Hint: start by showing that a positive maximum cannot be achieved in the interior or at the final boundary.)

(b) Show more generally that even if $M = \max_{x,t} c(x,t)$ is positive,

$$|u(x,t)| \le Ae^{Mt}$$

where A is the maximum of $|u|$ at the initial and spatial boundaries. (Hint: what differential equation does $e^{-Mt}u(x,t)$ satisfy?.)