**PDE for Finance Notes – Section 3 Addendum**

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**Verification**. In the deterministic case we used a heuristic argument to derive the HJB equation, but then showed completely honestly that a (sufficiently smooth) $C^1$ solution of the HJB equation (satisfying appropriate boundary or final-time conditions) provides a bound on the value attainable by any control. A similar result holds in the stochastic setting.

Rather than give a general result, let us focus on the example of Merton's optimal selection and consumption problem presented in Section 3. (The general result requires no additional ideas.) The state equation is

$$dy = [(1-\alpha_1)yr + \alpha_1 y\mu - \alpha_2]dt + \alpha_1 y\sigma dw$$

which we write for simplicity as

$$dy = f(y, \alpha_1, \alpha_2)dt + \alpha_1 y\sigma dw.$$

We defined the value function to be

$$u(x,t) = \max_{\alpha} E_{y(t)=x} \int_t^{\tau} e^{-\rho s} h[\alpha_2(s)]\, ds$$

where $\tau$ is either the first time $y=0$ (if this happens before time $T$) or $\tau = T$ (if $y$ doesn't reach 0 before time $T$). We derived the HJB equation:

$$u_t + \max_{a_1,a_2}\left\{e^{-\rho t}h(a_2) + f(x,a_1,a_2)u_x + \frac{1}{2}x^2 a_1^2\sigma^2 u_{xx}\right\} = 0$$

for $t < T$, with $u = 0$ at $t = T$. We didn't fuss over it before, but clearly $u$ should also satisfy $u(0,s) = 0$ for all $s$.

Consider any control $\tilde{\alpha}(s)$, and the associated state $\tilde{y}(s)$ starting from $\tilde{y}(t) = x$. Of course we assume $\tilde{\alpha}$ is non-anticipating, i.e. it depends only on knowledge of $\tilde{y}(s)$ in the present and past, not the future. We wish to show that

$$u(x,t) \geq E_{y(t)=x} \int_t^{\tilde{\tau}} e^{-\rho s} h[\tilde{\alpha}_2(s)]\, ds.$$

Consider $\phi(s) = u(\tilde{y}(s), s))$: by the Ito calculus it satisfies

$$\begin{aligned} d\phi &= u_s ds + u_y d\tilde{y} + \frac{1}{2}u_{yy} d\tilde{y} d\tilde{y} \\ &= u_s ds + u_y[f(\tilde{\alpha},\tilde{y})ds + \tilde{\alpha}_1(s)\tilde{y}(s)\sigma dw] + \frac{1}{2}u_{yy}\tilde{\alpha}_1^2(s)\tilde{y}^2(s)\sigma^2 ds. \end{aligned}$$

Therefore

$$u(\tilde{y}(t'),t') - u(\tilde{y}(t),t) = \int_t^{t'} [u_s + u_y f + \frac{1}{2}u_{yy}\tilde{y}^2\tilde{\alpha}_1^2\sigma^2]dt + \int_t^{t'} \sigma\tilde{\alpha}_1\tilde{y}u_y dw$$

where each integrand is evaluated at $y = \tilde{y}(s)$, $\alpha = \tilde{\alpha}(s)$ at time $s$. The expected value of the second integral is 0 (here is where we use that $\alpha$ is nonanticipating; this will be clearer after we discuss stochastic integrals, coming soon). Thus taking the expectation, and using the initial condition:

$$E\left[u(\tilde{y}(t'), t')\right] - u(x,t) = E\left[\int_t^{t'} (u_s + u_y f + \frac{1}{2} u_{yy} \tilde{y}^2 \tilde{\alpha}_1^2 \sigma^2) dt\right].$$

Now from the definition of the Hamiltonian we have

$$u_t(\tilde{y}(s), s) + \left\{e^{-\rho s} h(\tilde{\alpha}_2(s)) + f(\tilde{y}(s), \tilde{\alpha}(s)) u_y(\tilde{y}(s), s) + \frac{1}{2}\tilde{y}^2(s)\tilde{\alpha}_1^2(s)\sigma^2 u_{yy}(\tilde{y}(s), s)\right\} \leq 0.$$

Combining this with the preceding relation gives

$$E\left[u(\tilde{y}(t'), t')\right] - u(x,t) \leq -E\left[\int_t^{t'} e^{-\rho s} h(\tilde{\alpha}_2(s) ds\right].$$

Taking $t' = \tilde{\tau}$ and using the fact that $u(\tilde{y}(t'), t') = 0$, we conclude that

$$u(x,t) \geq E\left[\int_t^{\tilde{\tau}} e^{-\rho s} h(\tilde{\alpha}(s) ds\right]$$

as desired.

Notice that this calculation rests on pretty much the same tools we used to derive the HJB: (a) the Ito calculus, to get a representation of $u(\tilde{y}(s), s)$, and (b) the fact that the integral "$dw$" has expected value 0.