PDE for Finance, Spring 2000 – Homework 2 – distributed 2/8/00, due 2/22/00.

1) The Hopf-Lax solution formula solves the finite-horizon problem with state equation $dy/ds = \alpha$ and value function

$$u(x,t) = \max_{\alpha} \left\{ \int_t^T h(\alpha(s))\, ds + g(y(T)) \right\}$$

with h concave. The key step was to show that an optimal trajectory has constant velocity. Give an alternative justification of this fact using Pontryagin's maximum principle.

2) Finite-horizon problems and minimal-arrival-time problems are closely related. Let's explore the relation, then use it to deduce a version of Pontryagin's maximum principle for the minimum-time problem.

(a) Let $u(x)$ be value function of the minimum-time problem with target E, state equation $dy/ds = f(y,\alpha)$, and admissible controls $\alpha(s) \in A$. Let $v(x,t)$ be the value function of the finite horizon problem with the same state equation, the same set of admissible controls, and objective

$$\min_{\alpha} \int_t^T h_E(y(s))\, ds$$

where

$$h_E(x) = \begin{cases} 1 & \text{if } x \notin E \\ 0 & \text{if } x \in E. \end{cases}$$

Show that $v(x,t) = \min\{T-t, u(x)\}$.

(b) Use part (a) to deduce a version of the maximum principle for the minimum-time problem.

(c) Consider the special case when $f(y,\alpha) = \alpha$ and A is the unit sphere in R^n. This is Example 2 from the Section 1 notes, and we know that $u(x) = \text{dist}\,(x,E)$. What does your answer to (b) tell you about the optimal paths?

3) Example 3 in the Section 2 notes is a problem of optimal investment with proportional transaction costs. In formulating it we imposed a solvency constraint, but we permitted the investor to take a debt position in either instrument. Consider the analogous problem (same treatment of transaction costs) when investor is prohibited from taking a debt position in either instrument, i.e. he must keep $X(s) \geq 0$ and $Y(s) \geq 0$. The value function $u(x,y)$ is now defined just on the quadrant $x \geq 0, y \geq 0$.

(a) If the initial portfolio has $x = 0$, a plausible strategy is to keep $x = 0$ forever, transfering just enough funds from high-yield to money-market to cover your consumption. What formula does this suggest for $u(0,y)$? [Hint: this is related to problem 2 of HW1.]

(b) If the initial portfolio has $y = 0$, a plausible strategy is to transfer funds immediately from money-market to high-yield so that (after the transfer) $x = h_0 y$ for some constant h_0. Homogeneity tells us that $u(x, 0) = c_0 x^p$ and $u(h_0 y, y) = c_1 y^p$ for some constants c_0 and c_1. Assuming the proposed strategy is optimal, what is the relation between c_0 and c_1?

4) Example 3 assumes proportional transaction costs: of every dollar transfered, fraction μ goes to commissions and fraction $1 - \mu$ arrives at the destination. What if transaction costs are not proportional? For example, what if the commission is 10% for transfers up to \$1000, but only 5% on the excess over \$1000. (Thus the commission on a transfer of \$1500 is $.1 \times 1000 + .05 \times 500 = 125$.) Can you suggest a method for modeling this? [Hint: try making time discrete.]

5) We used the Hopf-Lax solution formula to see that the dynamic programming solution of $u_t + \frac{1}{2}u_x^2 = 0$ with $u = |x|$ at $t = T$ is $u(x, t) = (T - t) + |x|$. What happens when we change the final-time condition to

$$u = \begin{cases} \frac{1}{\epsilon}x^2 & \text{if } |x| \leq \epsilon/2 \\ |x| - \frac{\epsilon}{4} & \text{if } |x| \geq \epsilon/2, \end{cases}$$

at $t = T$. (This is a C^1 approximation to $|x|$.) Does the resulting solution have continuous derivatives, or does its graph still have a sharp valley?

6) The Section 1 notes give a "minimum travel time" problem whose value function u solves $|\nabla u| = 1$ for $x \notin E$ with $u = 0$ at ∂E.

(a) Find a related dynamic programming problem whose value function (if smooth) should solve $|\nabla u| = 1$ for $x \notin E$ with $u = g$ at ∂E, where g is a specified function on ∂E.

(b) Consider the 2D case, with E a planar region with smooth boundary ∂E. Describe the optimal controls and paths, if g is smooth and its derivative (with respect to arc-length) on E satisfies $|g'| < 1$.

(c) What changes if $|g'|$ is bigger than 1 on some part of ∂E?

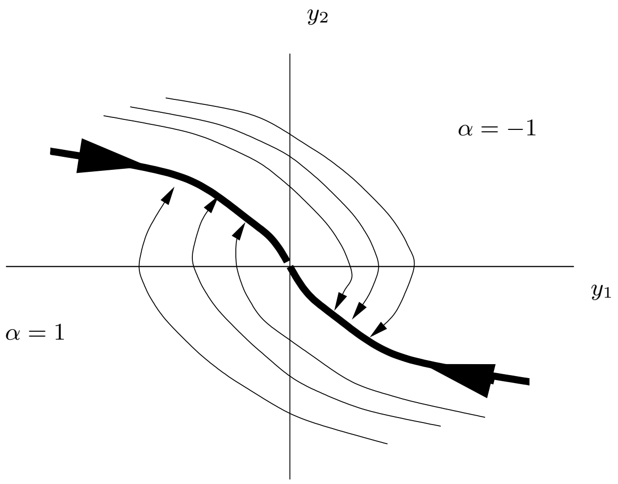
7) Consider the following physically natural minimum-time problem (sometimes known as the "rocket-car problem"). A 1D particle with mass 1 has position x_1 and velocity x_2 at time 0. You can control it by applying a force of magnitude less then or equal to 1. Your goal is to bring it to rest at the origin as quickly as possible.

(a) Show we are considering a minimum-time problem with dynamics

$$dy_1/ds = y_2, \quad dy_2/ds = \alpha(s),$$

control $\alpha(s) \in A = \{|a| \leq 1\}$ and target set $\mathcal{T} = \{0, 0\}$.

(b) Find the associated Hamilton-Jacobi-Bellman equation.



(c) Show that when $a = 1$ the state moves along one of the parabolas $y_1 = \frac{1}{2}y_2^2 + c$. Similarly, if $a = -1$ the state moves along one of the parabolas $y_1 = -\frac{1}{2}y_2^2 + c$. From which starting points can the state move along one of these parabolas and arrive at $y_1 = y_2 = 0$?

(d) Show the following "feedback control" drives any initial state (x_1, x_2) to $(0, 0)$: take $\alpha(s)$ to be the following function of the state $(y_1(s), y_2(s))$:

$$\alpha = \begin{cases} -1 & \text{if } y_1 > -\frac{1}{2}y_2|y_2| \\ 1 & \text{if } y_1 > 0 \text{ and } y_1 = -\frac{1}{2}y_2|y_2| \\ 1 & \text{if } y_1 < -\frac{1}{2}y_2|y_2| \\ -1 & \text{if } y_1 < 0 \text{ and } y_1 = -\frac{1}{2}y_2|y_2| \end{cases}$$

(See the figure to visualize this; apologies for its crudeness.)

Show moreover that this control achieves value

$$u(x) = \begin{cases} x_2 + 2(x_1 + x_2^2/2)^{1/2} & \text{if } x_1 \geq -\frac{1}{2}x_2|x_2| \\ -x_2 + 2(-x_1 + x_2^2/2)^{1/2} & \text{if } x_1 \leq -\frac{1}{2}x_2|x_2| \end{cases}$$

(e) Show by a suitable verification argument that the control specified in (d) is optimal. (Hint: show, as a first step, that it is optimal if (x_1, x_2) happens to lie on the curve $x_1 = -\frac{1}{2}x_2|x_2|$.)

8) This problem is a special case of the "linear-quadratic regulator" widely used in engineering applications. The state is $y(s) \in R^n$, and the control is $\alpha(s) \in R^n$. There is no pointwise restriction on the values of $\alpha(s)$. The evolution law is

$$dy/ds = Ay + \alpha, \quad y(t) = x,$$

for some constant matrix A, and the goal is to minimize

$$\int_t^T |y(s)|^2 + |\alpha(s)|^2 \, ds + |y(T)|^2.$$

(In words: we prefer $y = 0$ along the trajectory and at the final time, but we also prefer not to use too much control.)

(a) What is the associated Hamilton-Jacobi-Bellman equation? Explain why we should expect the relation $\alpha(s) = -\frac{1}{2}\nabla u(y(s))$ to hold along optimal trajectories.

(b) Since the problem is quadratic, it's natural to guess that the value function $u(x,t)$ takes the form

$$u(x,t) = \langle K(t)x, x\rangle$$

for some symmetric $n \times n$ matrix-valued function $K(t)$. Show that this u solves the Hamilton-Jacobi-Bellman equation exactly if

$$\frac{dK}{dt} = K^2 - I - (K^T A + A^T K) \text{ for } t < T, \quad K(T) = I$$

where I is the $n \times n$ identity matrix. (Hint: two quadratic forms agree exactly if the associated symmetric matrices agree.)

(c) Show by a suitable verification argument that this u is indeed the value function of the control problem.