PDE for Finance, Spring 2000 – Homework 4
Distributed 3/7/00, due 3/21/00. Solutions will be distributed 3/28/00.

1) Problem 4 of HW3 considered the stochastic "linear quadratic regulator" problem in continuous time. Here is the analogous stochastic discrete-time problem. We label times by $k = 0, 1, \ldots$. The state at time k is $y_k \in R^n$, and the control at time k is $\alpha_k \in R^n$. We place no restriction on the possible values of α_k. The state equation is

$$y_{k+1} = Ay_k + \alpha_k + e_k$$

where A is a (known) $n \times n$ matrix, and the e_k's are independent, identically distributed random variables with mean value 0 and finite variance. We emphasize that e_k is independent of y_k and α_k. The initial condition is $y_0 = x$, and the goal is to minimize the expected cost

$$E_{y_0=x}\left\{\sum_{k=0}^{N-1}[|y_k|^2 + |\alpha_k|^2] + |y_N|^2\right\}.$$

The interpretation is the same as in the continuous case: we prefer $y = 0$. The system keeps getting perturbed away from this state by noise; the control must be chosen to bring it back, but there is also a cost associated to the control itself.

Let $J_k(x)$ be the minimum expected cost if the initial stage is k and the initial state is x. Observe that $J_N(x) = |x|^2$.

(a) Write the dynamic programming relation connecting J_k to J_{k+1}.

(b) Look for a solution of the form $J_k(x) = \langle K_k x, x\rangle + q_k$, where K_k is a symmetric matrix and q_k is a scalar. Show that K_k satisfies the following recurrence relation:

$$K_k = A^T\left[K_{k+1} - K_{k+1}(K_{k+1} + I)^{-1}K_{k+1}\right]A + I$$

with $K_N = I$. How is (the optimal) α_k related to y_k? What is the value of q_k?

(Remark: For much more about the discrete-time LQR problem see section 2.1 of Bertsekas.)

2) This problem develops a continuous-time analogue of the simple Bertsimas & Lo model of "Optimal control of execution costs" presented in the Section 4 notes. The state is (w, p), where w is the number of shares yet to be purchased and p is the current price per share. The control $\alpha(s)$ is the rate at which shares are purchased. The state equation is:

$$\begin{aligned} dw &= -\alpha\, ds \text{ for } t < s < T, \quad w(t) = w_0 \\ dp &= \theta\alpha\, ds + \sigma dz \text{ for } t < s < T, \quad p(t) = p_0 \end{aligned}$$

where dz is Brownian motion and θ, σ are fixed constants. The goal is to minimize, among (nonanticipating) controls $\alpha(s)$, the expected cost

$$E\left\{\int_t^T [p(s)\alpha(s) + \theta\alpha^2(s)]\, ds + [p(T)w(T) + \theta w^2(T)]\right\}.$$

The optimal expected cost is the value function $u(w_0, p_0, t)$.

(a) Show that the HJB equation for u is

$$u_t + H(u_w, u_p, p) + \frac{\sigma^2}{2} u_{pp} = 0$$

for $t < T$, with Hamiltonian

$$H(u_w, u_p, p) = -\frac{1}{4\theta}(p + \theta u_p - u_w)^2.$$

The final value is of course

$$u(w, p, T) = pw + \theta w^2.$$

(b) Look for a solution of the form $u(w, p, t) = pw + g(t)w^2$. Show that g solves

$$\dot{g} = \frac{1}{4\theta}(\theta - 2g)^2$$

for $t < T$, with $g(T) = \theta$. Notice that u does not depend on σ, i.e. setting $\sigma = 0$ gives the same value function.

(c) Solve for g. (Hint: start by rewriting the equation for g, "putting all the g's on the left and all the t's on the right".)

(d) Show by direct examination of your solution that the optimal $\alpha(s)$ is constant.

(e) Give another proof that the optimal $\alpha(s)$ is constant, by examining the deterministic version of this control problem ($\sigma = 0$) and arguing roughly as we did for the Hopf-Lax solution formula (using the convexity of α^2).

(Remark: a better choice of objective would be $E\left\{\int_t^T [p(s)\alpha(s) + \theta'\alpha^2(s)]\, ds + [p(T)w(T) + \theta'' w^2(T)]\right\}$ for some constants θ', θ'', since the state equation gives θ units of dollars/(share)^2, whereas the units of θ' and θ'' are different. Food for thought: what happens if one takes the running cost to be $\int_t^T p(s)\alpha(s)\, ds$ instead of $\int_t^T p(s)\alpha(s) + \theta\alpha^2(s)\, ds$?)

3) [from Bertsekas: chapter 2, problem 12]. A gambler plays a game in which he may at each time k stake any amount $u_k \geq 0$ that does not exceed his current fortune x_k (defined to be his initial capital plus his gain or minus his loss thus far). He wins his stake back and as much more with probability p, where $\frac{1}{2} < p < 1$, and he loses his stake with probability $(1 - p)$. His goal is to maximize $E\{\log x_N\}$, where x_N is his fortune after N plays. Let's give two separate proofs that his optimal policy is to stake, at each play, $2p - 1$ times his current fortune (i.e. to choose $u_k = (2p - 1)x_k$).

(a) Let x_0 be the gambler's initial capital, and let $q_k = u_k/x_k$ be the fraction of his wealth he stakes at time k. His *return* at time k is

$$R_k = \begin{cases} (1 + q_k) & \text{with probability } p \\ (1 - q_k) & \text{with probability } 1 - p \end{cases}$$

in the sense that $x_{k+1} = R_k x_k$. It follows that

$$\log x_N = \log x_0 + \log R_0 + \ldots + \log R_{N-1},$$

whence

$$E[\log x_N] = \log x_0 + E[\log R_0] + \ldots + E[\log R_{N-1}].$$

Show that $E[\log R_k]$ is maximized, for each k, by the choice $q_k = 2p - 1$.

(b) Give an alternative analysis based on the principle of dynamic programming. Use $J_k(x_k) = E[\log x_N]$ as your value function, where k is the current time, x_k is the current wealth, and the expectation refers to all remaining uncertainty (the outcome of betting at times $k, \ldots, N-1$).

[Remark: the first approach works – i.e. the method of dynamic programming isn't really needed here – because the optimal policy is "myopic," i.e. it optimizes each time step separately. This is a special to the use of $\log x_N$ as the objective.]

4) [from Bertsekas: chapter 2, problem 19]. A driver is looking for a parking place on the way to his destination. Each parking place is free with probability p, independent of whether other parking spaces are free or not. The driver cannot observe whether a parking place is free until he reaches it. If he parks k places from his destination, he incurs a cost k. If he reaches the destination without having parked the cost is C.

(a) Let F_k be the minimal expected cost if he is k parking places from his destination. Show that

$$\begin{aligned} F_0 &= C \\ F_k &= p \min[k, F_{k-1}] + qF_{k-1}, \quad k = 1, 2, \ldots \end{aligned}$$

where $q = 1 - p$.

(b) Show that an optimal policy is of the form: never park if $k \geq k^*$, but take the first free place if $k < k^*$, where k is the number of parking places from the destination, and k^* is the smallest integer i satisfying $q^{i-1} < (pC + q)^{-1}$.

5) The Section 4 notes discuss work by Bertsimas, Logan, and Lo involving least-square replication of a European option. The analysis there assumes all trades are *self-financing*, so the value of the portfolio at consecutive times is related by

$$V_j - V_{j-1} = \theta_{j-1}(P_j - P_{j-1}).$$

Let's consider what happens if trades are permitted to be non-self-financing. This means we introduce an additional control g_j, the amount of cash added to (if $g_j > 0$) or removed from (if $g_j < 0$) the portfolio at time j, and the portfolio values now satisfy

$$V_j - V_{j-1} = \theta_{j-1}(P_j - P_{j-1}) + g_{j-1}.$$

It is natural to add a quadratic expression involving the g's to the objective: now we seek to minimize

$$E\left[(V_N - F(P_N))^2 + \alpha \sum_{j=0}^{N-1} g_j^2\right]$$

where α is a positive constant. The associated value function is

$$J_i(V,P) = \min_{\theta_i, g_i; \ldots; \theta_{N-1}, g_{N-1}} E_{V_i=V,\, P_i=P}\left[(V_N - F(P_N))^2 + \alpha \sum_{j=i}^{N-1} g_j^2\right].$$

The claim enunciated in the Section 4 notes remains true in this modified setting: J_i can be expressed as a quadratic polynomial

$$J_i(V_i, P_i) = \bar{a}_i(P_i)|V_i - \bar{b}_i(P_i)|^2 + \bar{c}_i(P_i)$$

where $\bar{a}_i, \bar{b}_i$, and $\bar{c}_i$ are suitably-defined functions which can be constructed inductively. Demonstrate this assertion in the special case $i = N-1$, and explain how $\bar{a}_{N-1}, \bar{b}_{N-1}, \bar{c}_{N-1}$ are related to the functions $a_{N-1}, b_{N-1}, c_{N-1}$ of the Section 4 notes.