PDE for Finance, Spring 2000 – Homework 6
Distributed 4/18/00, due 4/25/00. Solutions will be distributed 5/2/00.

Reminders:

- The last class is Tuesday May 2.
- The final exam is Tuesday May 9. You may bring two sheets of notes – 8.5×11, both sides, write as small as you like.

1) Consider linear heat equation $u_t - u_{xx} = 0$ in one space dimension, with discontinuous initial data

$$u(x,0) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0. \end{cases}$$

(a) Show by evaluating the solution formula that

$$u(x,t) = \frac{1}{2}\left[1 + \phi(x/\sqrt{4t})\right]$$

where ϕ is the "error function"

$$\phi(s) = \frac{2}{\sqrt{\pi}} \int_0^s e^{-r^2}\, dr.$$

(b) Explore the solution by answering the following: what is $\max_x u_x(x,t)$ as a function of time? Where is it achieved? What is $\min_x u_x(x,t)$? For which x is $u_x > (1/10)\max_x u_x$? Sketch the graph of u_x as a function of x at a given time $t > 0$.

(c) Show that $v(x,t) = \int_{-\infty}^x u(z,t)\, dz$ solves $v_t - v_{xx} = 0$ with $v(x,0) = \max\{x, 0\}$. Deduce the qualitative behavior of $v(x,t)$ as a function of x for given t: how rapidly does v tend to 0 as $x \to -\infty$? What is the behavior of v as $x \to \infty$? What is the value of $v(0,t)$? Sketch the graph of $v(x,t)$ as a function of x for given $t > 0$.

[Comment: this problem is intended to give intuition concerning the value near maturity of a European call.]

2) This problem can be done in any space dimension, but we stick to 1D for simplicity. Consider the stochastic differential equation $dy = f(y,s)ds + g(y,s)dw$, and the associated backward and forward Kolmogorov equations

$$u_t + f(x,t)u_x + \tfrac{1}{2}g^2(x,t)u_{xx} = 0 \quad \text{for } t < T, \text{ with } u = \Phi \text{ at } t = T$$

and

$$\rho_s + (f(z,s)\rho)_z - \tfrac{1}{2}(g^2(z,s)\rho)_{zz} = 0 \quad \text{for } s > 0, \text{ with } \rho(z) = \rho_0(z) \text{ at } s = 0.$$

Recall that $u(x,t)$ is the expected value (starting from x at time t) of payoff $\Phi(y(T))$, whereas $\rho(z,s)$ is the probability distribution of the diffusing state $y(s)$ (if the initial distribution is ρ_0).

(a) The solution of the backward equation has the following property: if $m = \min_z \Phi(z)$ and $M = \max_z \Phi(z)$ then $m \leq u(x,t) \leq M$ for all $t < T$. Give two distinct justifications: one using the maximum principle for the PDE, the other using the probabilistic interpretation.

(b) The solution of the forward equation does *not* in general have the same property; in particular, $\max_z \rho(z,s)$ can be larger than the maximum of ρ_0. Explain why, by considering the example $dy = -yds$. (Intuition: $y(s)$ moves toward the origin; in fact, $y(s) = e^{-s}y_0$. Viewing $y(s)$ as the position of a moving particle, we see that particles tend to collect at the origin no matter where they start. So $\rho(z,s)$ should be increasingly concentrated at $z = 0$.) Show that the solution in this case is $\rho(z,s) = e^s \rho_0(e^s z)$. This example has $g = 0$; can you suggest what would happen when $g = \epsilon$, a sufficiently small constant?

3) The solution of the forward Kolmogorov equation is a probability density, so we expect it to be nonnegative (assuming the initial condition $\rho_0(z)$ is everywhere nonnegative). In light of Problem 2b it's natural to worry whether the PDE has this property. Let's show that it does.

(a) Consider the initial-boundary-value problem

$$w_t = a(x,t)w_{xx} + b(x,t)w_x + c(x,t)w$$

with x in the interval $(0,1)$ and $0 < t < T$. We assume as usual that $a(x,t) > 0$. Suppose furthermore that $c < 0$ for all x and t. Show that if $0 \leq w \leq M$ at the initial time and the spatial boundary then $0 \leq w \leq M$ for all x and t. (Hint: a positive maximum cannot be achieved in the interior or at the final boundary. Neither can a negative minimum.)

(b) Now consider the same PDE but with $\max_{x,t} c(x,t)$ positive. Suppose the initial and boundary data are nonnegative. Show that the solution w is nonnegative for all x and t. (Hint: apply part (a) not to w but rather to $\bar{w} = e^{-Ct}w$ with a suitable choice of C.)

(c) Consider the solution of the forward Kolmogorov equation in the interval, with $\rho = 0$ at the boundary. (It represents the probability of arriving at z at time s without hitting the boundary first.) Show using part (b) that $\rho(z,s) \geq 0$ for all s and z. What is the condition on f and g that $\max_z \rho(z,s) \leq \max \rho_0$? How does this condition generalize to the multidimensional case?

[Comment: statements analogous to (a)-(c) are valid for the initial-value problem as well, when we solve for all $x \in R$ rather than for x in a bounded domain. The justification takes a little extra work however, and it requires some hypothesis on the growth of the solution at ∞.]

4) Consider the solution of

$$u_t + au_{xx} = 0 \quad \text{for } t < T, \text{ with } u = \Phi \text{ at } t = T$$

where a is a positive constant. Recall that in the stochastic interpretation, a is $\frac{1}{2}g^2$ where g represents volatility. Let's use the maximum principle to understand qualitatively how the solution depends on volatility.

(a) Show that if $\Phi_{xx} \geq 0$ for all x then $u_{xx} \geq 0$ for all x and t.

(b) Suppose $\bar{u}$ solves the analogous equation with a replaced by $\bar{a} > a$, using the same final-time data Φ. We continue to assume that $\Phi_{xx} \geq 0$. Show that $\bar{u} \geq u$ for all x and t. (Hint: $w = \bar{u} - u$ solves $w_t + \bar{a}w_{xx} = f$ with $f = (a - \bar{a})u_{xx} \leq 0$.)

(c) Do the conclusions of (a) and (b) remain valid when volatility is non-constant, i.e. $a = a(x,t)$ is a deterministic function of x and t?

[Comment: The Black-Scholes PDE for options on a lognormal asset can be reduced by change of variables to the linear heat equation. For a call, the relevant choice of Φ has the form $\Phi(x) = \max\{e^{\alpha x} - e^{(\alpha-1)x}, 0\}$ for a suitable choice of $\alpha > 0$. This problem shows that increasing the volatility of the underlying asset increases the value of the call. The same argument does not work for a put – why not?]

5) Early this semester we encountered the Hopf-Lax solution formula for the HJ equation $u_t + H(\nabla u) = 0$ with H is convex, and we discussed at length the example

$$u_t + \tfrac{1}{2}u_x^2 = 0 \quad \text{for } t < T \text{ with } u = \Phi \text{ at } t = T,$$

for which the Hopf-Lax formula gives

$$u(x,t) = \max_z \{\Phi(z) - \frac{|z-x|^2}{2(T-t)}\}.$$

The PDE has many "almost-everywhere" solutions, but the one given by the Hopf-Lax formula is special: it (a) gives the value function for an associated control problem, and (b) gives the "viscosity solution" of the HJ equation. Restatement of the latter: it is the solution obtained by solving

$$u_t + \tfrac{1}{2}u_x^2 + \epsilon u_{xx} = 0 \quad \text{for } t < T \text{ with } u = \Phi \text{ at } t = T,$$

with $\epsilon > 0$, then taking the limit $\epsilon \to 0$. Let's explore a trick which makes this limit (in this special case) very explicit:

(a) Consider the function $w = e^{u/2\epsilon}$. Show that $w_t + \epsilon w_{xx} = 0$ with $w = e^{\Phi/2\epsilon}$ at $t = T$.

(b) Deduce the solution formula:

$$e^{u(x)/2\epsilon} = \frac{1}{\sqrt{4\pi\epsilon(T-t)}} \int_{-\infty}^{\infty} e^{\frac{1}{2\epsilon}\left[\Phi(z) - \frac{|x-z|^2}{2(T-t)}\right]}\, dz$$

[Comment: One can show that this reduces to the Hopf-Lax formula in the limit $\epsilon \to 0$. The idea is easy, though the details take some work: when ϵ is small, the integral is dominated by the z's where $\Phi(z) - \frac{|x-z|^2}{2(T-t)}$ is largest. The change of variables $w = e^{u/2\epsilon}$ is known as the Hopf-Cole transformation.]

6) Consider the standard finite difference scheme

$$\frac{u((m+1)\Delta t, n\Delta x) - u(m\Delta t, n\Delta x)}{\Delta t} = \frac{u(m\Delta t, (n+1)\Delta x) - 2u(m\Delta t, n\Delta x) + u(m\Delta t, (n-1)\Delta x)}{(\Delta x)^2} \tag{1}$$

for solving $u_t - u_{xx} = 0$. The stability restriction $\Delta t < \frac{1}{2}\Delta x^2$ leaves a lot of freedom in the choice of Δx and Δt. Show that

$$\Delta t = \frac{1}{6}\Delta x^2$$

is special, in the sense that the numerical scheme (1) has errors of order Δx^4 rather than Δx^2. In other words: when u is the exact solution of the PDE, the left and right sides of (1) differ by a term of order Δx^4. [Comment: the argument sketched at the end of the Section 6 notes shows that if u solves the PDE and v solves the finite difference scheme then $|u - v|$ is of order Δx^2 in general, but it is smaller – of order Δx^4 – when $\Delta t = \frac{1}{6}\Delta x^2$.]