PDE for Finance Notes – Section 8

Notes by Robert V. Kohn, Courant Institute of Mathematical Sciences. For use only in connection with the NYU course PDE for Finance, G63.2706, Spring 2000.

Reminder concerning the final: The exam will be Tuesday May 9, at the usual class time. It will be "closed-book" (no books, no lecture notes), however you may bring two sheets of notes (8.5×11, both sides, write as small as you like). You are responsible for the material in Sections 1-6 of the lecture notes, and in Homeworks 1-6. See the separate handout for further discussion of what to expect. (The material in these Section 8 notes will not be on the exam.)

****

Some explicit solution formulas for the constant-coefficient heat equation in one space dimension. The Black-Scholes PDE can be reduced by change of variables to the constant-coefficient linear heat equation. (This was essentially a homework problem, for the simplest case – when the underlying asset has constant volatility and the risk-free rate is constant. There is a similar reduction when the volatility and risk-free rate are deterministic functions of time. The crucial hypotheses are that these functions are known in advance, and that the volatility is not a function of stock price.) Therefore explicit solution formulas for the 1D linear heat equation are useful for pricing options.

We discussed in Section 6 the case when x ranges over the entire real line:

(a) The WHOLE-SPACE PROBLEM

$$u_t = u_{xx} \quad \text{for } t > 0, \text{ with } u = g \text{ at } t = 0.$$

The solution to (a) is

$$u(x,t) = \int_{-\infty}^{\infty} k(x-y,t)g(y)\,dy \tag{1}$$

where

$$k(z,t) = \frac{1}{\sqrt{4\pi t}} e^{-z^2/4t} \tag{2}$$

This leads (using the change of variables) to an explicit formula for value of a (vanilla) European option with any payoff. Notice that $k(z,t)$ gives the probability of a Brownian walker being at z at time t, if it started at the origin at time 0. This k is called the *fundamental solution* of the heat equation. (The statement I just made about the "Brownian walker" was sloppy: it would have been true if we had solved $u_t = \frac{1}{2}u_{xx}$. In the present context the relevant random walk is $\sqrt{2}$ times Brownian motion.)

There are similarly explicit solution formulas in several other cases:

(b) The WHOLE-SPACE PROBLEM WITH A SOURCE

$$u_t = u_{xx} + f(x,t) \quad \text{for } t > 0, \text{ with } u = g \text{ at } t = 0. \tag{3}$$

Recall from our discussion of the backward Kolmogorov equation that a running term in the payoff shows up as a source term in the equation. In pricing options, a source term can arise from dividends.

(c) The INITIAL-VALUE PROBLEM FOR A HALF-SPACE

$$u_t = u_{xx} \quad \text{for } t > 0 \text{ and } x > x_0\text{, with } u = g \text{ at } t = 0 \text{ and } u = \phi \text{ at } x = x_0. \tag{4}$$

Since this is a boundary-value problem, we must specify data both at the initial time $t = 0$ and at the spatial boundary $x = 0$. We arrived at this type of problem in our discussion of the backward Kolmogorov equation when we considered a payoff defined at an exit time. The relevant option-pricing problems involve barriers. If the option becomes worthless at when the stock price crosses the barrier then $\phi = 0$ (this is a knock-out option). If the option turns into a different instrument when the stock price crosses the barrier then ϕ is the value of that instrument.

These notes explain the solution formulas for problems (b) and (c).

The whole-space problem with a source. The heat equation is a PDE, but it's sometimes convenient to think of it as an "infinite-dimensional ODE." To explain this viewpoint, consider first the simple ODE

$$\frac{dz}{dt} = Az + f. \tag{5}$$

when A is a *constant* $n \times n$ matrix and $f = (f_1, \ldots, f_n)$ is a known function of time. We must also specify the initial condition $z(0)$; then the equation determines the future values $z = (z_1, \ldots, z_n)$ as a function of time. Because A is constant the equation is easy to solve: multiplying both sides by e^{-At} and doing a bit of calculation we see that

$$\frac{d}{dt}\left[e^{-At}z\right] = e^{-At}f$$

whence

$$e^{-At}z(t) - z(0) = \int_0^t e^{-As}f(s)\,ds$$

and it follows that

$$z(t) = e^{At}z(0) + \int_0^t e^{A(t-s)}f(s)\,ds.$$

Notice that the first term gives the solution when $f = 0$ (in other words $z(t) = e^{At}z(0)$ solves $dz/dt = Az$ with initial condition $z(0)$). The second term gives the solution when $z(0) = 0$ (in other words $z(t) = \int_0^t e^{A(t-s)}f(s)\,ds$ solves $dz/dt = Az + f$ with initial condition 0). It is natural for the solution to come in this two-part form, since the problem is *linear*.

What does this have to do with the PDE $u_t = \Delta u + f$? One way to see that the PDE resembles an ODE is to consider our explicit finite difference scheme with $\Delta t = 0$. It gives an ODE for the values of u at grid points, $u(t, j\Delta x)$. If space is bounded then this is quite literally an ODE of the type considered above. This example suggests the correct viewpoint: we should view $t \to u(t, \cdot)$ as a function of time taking values in the infinite-dimensional vector space of functions of x, and we should view the Laplacian Δ as a linear operator on functions of x. In summary: the heat equation is like the ODE (5), with $u(t,\cdot)$ playing the role of $z(t)$ and $\Delta u = u_{xx}$ playing the role of Az. The source term is f in both cases, but for the PDE we view it as $t \to f(t,\cdot)$, a function of time whose values are functions of space.

This (formal, but justifiable) analogy suggests the following solution formula for (3):

$$u(t) = e^{t\Delta} g + \int_0^t e^{(t-s)\Delta} f(s)\, ds. \tag{6}$$

This is correct, provided we interpret it properly. Both sides are functions of space at time t. The value of the left hand side at x is $u(x,t)$. The function $e^{t\Delta} g$ is the solution of the heat equation $u_t = u_{xx}$ with initial data g at time 0, so

$$e^{t\Delta} g = \int_{-\infty}^{\infty} k(x-y,t) g(y)\, dy$$

where k is the fundamental solution. The function $e^{(t-s)\Delta} f(s)$ is similarly the solution of the heat equation with initial data $f(s)$ at time s, evaluated at the later time $t = s + (t-s)$; we know this is

$$e^{(t-s)\Delta} f(s) = \int_{-\infty}^{\infty} k(x-y,t-s) f(y,s)\, dy.$$

Thus interpreted, right hand side of (6) is indeed the solution of (3).

The half-space problem with boundary condition 0**.** It's clear, by linearity, that the solution of (4) can be written as $u = v + w$, where v solves

$$v_t = v_{xx} \quad \text{for } t > 0 \text{ and } x > x_0, \text{ with } v = g \text{ at } t = 0 \text{ and } v = 0 \text{ at } x = x_0 \tag{7}$$

(in other words: v solves the same PDE with the same initial data but boundary data 0) and w solves

$$w_t = w_{xx} \quad \text{for } t > 0 \text{ and } x > x_0, \text{ with } w = 0 \text{ at } t = 0 \text{ and } w = \phi \text{ at } x = x_0 \tag{8}$$

(in other words: w solves the same PDE with the same boundary data but initial data 0). There is no loss of generality in taking $x_0 = 0$, and we make this choice henceforth.

We concentrate for the moment on v. To obtain its solution formula, we consider the whole-space problem with the *odd reflection of* g as initial data. Remembering that $x_0 = 0$, this odd reflection is defined by

$$\tilde{g}(x) = \begin{cases} g(x) & \text{if } x > 0 \\ -g(-x) & \text{if } x < 0 \end{cases}$$

(see Figure 1). Notice that the odd reflection is continuous at 0 if $g(0) = 0$; otherwise it is discontinuous, taking values $\pm g(0)$ just to the right and left of 0.

Let $\tilde{v}(x,t)$ solve the whole-space initial-value problem with initial condition $\tilde{g}$. We claim

- $\tilde{v}$ is a smooth function of x and t for $t > 0$ (even if $g(0) \neq 0$);

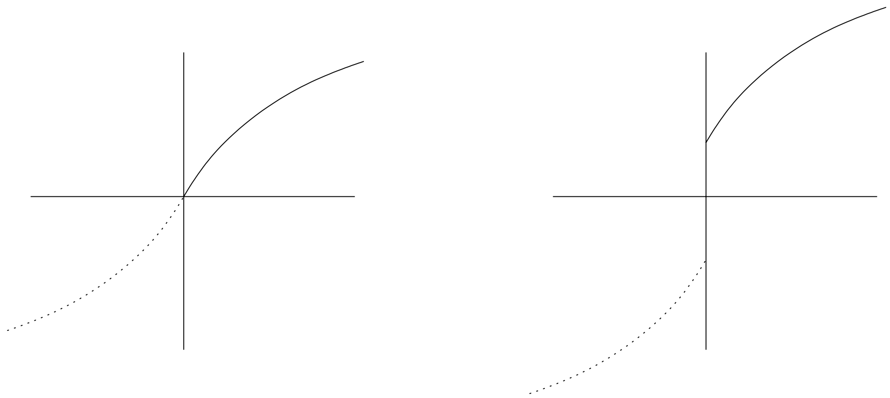

Figure 1: *Odd reflection. Note that the odd reflection is discontinuous at* 0 *if the original function doesn't vanish there.*

- $\tilde{v}(x,t)$ is an odd function of x for all t, i.e. $\tilde{v}(x,t) = -\tilde{v}(-x,t)$.

The first bullet follows from the smoothing property of the heat equation. The second bullet follows from the uniqueness of solutions to the heat equation, since $\tilde{v}(x,t)$ and $-\tilde{v}(-x,t)$ both solve the heat equation with the *same* initial data $\tilde{g}$.

We're essentially done. The oddness of $\tilde{v}$ gives $\tilde{v}(0,t) = -\tilde{v}(0,t)$, so $\tilde{v}(0,t) = 0$ for all $t > 0$. Thus

$$v(x,t) = \tilde{v}(x,t), \quad \text{restricted to } x > 0$$

is the desired solution to (7). Of course it can be expressed using (1): a formula encapsulating our solution procedure is

$$\begin{aligned} v(x,t) &= \int_0^\infty k(x-y,t)g(y)\,dy + \int_{-\infty}^0 k(x-y,t)(-g(-y))\,dy \\ &= \int_0^\infty [k(x-y,t) - k(x+y,t)]g(y)\,dy \end{aligned}$$

where $k(x,t)$ is the fundamental solution of the heat equation, given by (2). Notice that

$$v(x,t) = \int_0^\infty G(x,y,t)g(y)\,dy$$

with

$$G(x,y,s) = k(x-y,t) - k(x+y,t). \tag{9}$$

Notice that $G(x,y,s) = G(y,x,s)$ so we don't have to try to remember which variable (x or y) we put first. The function G is called the "Green's function" of the half-space problem. Based on our discussion of the forward Kolmogorov equation, we recognize $G(x,y,t)$ as giving the probability that a Brownian particle starting from y at time 0 reaches position x at time t without first reaching the origin. (Here again I'm being sloppy: the relevant random walk is not Brownian motion but $\sqrt{2}$ times Brownian motion.)

Remark: notice that it makes a great deal of difference whether $g(0)$ vanishes or not. If $g(0) \neq 0$ then the solution v has strange behavior at $x = t = 0$, since it vanishes when we approach this point along the spatial boundary ($x = 0$, $t > 0$) but not when we approach it along the initial boundary ($t = 0$, $x > 0$). Such behavior occurs, for example, when pricing a knock-out barrier option, if the barrier is to the wrong side of the strike price.

The half-space problem with initial condition 0**.** It remains to consider w, defined by (8). It solves the heat equation on the half-space, with initial value 0 and boundary value $\phi(t)$. We focus on the case when the ϕ is *compatible* with the initial data in the sense that

$$\phi(0) = 0 \tag{10}$$

so that w is continuous at $x = 0$, $t = 0$. The solution w is given by

$$w(x,t) = \int_0^t \frac{\partial G}{\partial y}(x, 0, t-s)\phi(s)\, ds \tag{11}$$

where $G(x, y, t)$ is the Green's function of the half-space problem given by (9). Using the formula derived earlier for G, this amounts to

$$w(x,t) = \int_0^t \frac{x}{(t-s)\sqrt{4\pi(t-s)}} e^{-x^2/4(t-s)}\phi(s)\, ds$$

The justification of (11) is not difficult, but it's rather different from what we've done before. Consider the function ψ which solves the heat equation *backward in time* from time t, with final-time data concentrated at x_0 at time t (use Figure 2 to visualize the geometry). We mean ψ to be defined only for $x > 0$, with $\psi = 0$ at the spatial boundary $x = 0$. In formulas, our definition is

$$\psi_\tau + \psi_{yy} = 0 \quad \text{for } \tau < t \text{ and } y > 0, \text{ with } \psi = \delta_{x_0} \text{ at } \tau = t \text{ and } \psi = 0 \text{ at } y = 0.$$

A formula for ψ is readily available, since the change of variable $s = t - \tau$ transforms the problem solved by ψ one considered earlier for v:

$$\psi(y, \tau) = G(x_0, y, t - \tau). \tag{12}$$

What's behind our strange-looking choice or ψ? Two things. First, the choice of final-time data gives

$$w(x_0, t) = \int \psi(y,t) w(y,t)\, dy.$$

(The meaning of the statement "$\psi = \delta_{x_0}$ at time t" is precisely that this holds for every continuous w). Second, if w solves the heat equation forward in time and ψ solves it backward in time then

$$\begin{aligned}
\frac{d}{ds}\int_0^\infty \psi(y,s)w(y,s)\, dy &= \int_0^\infty \psi_s w + \psi w_s\, dy \\
&= \int_0^\infty -\psi_{yy} w + \psi w_{yy}\, dy \\
&= \int_0^\infty -(\psi_y w)_y + (\psi w_y)_y\, dy \\
&= \left(-\psi_y w + \psi w_y\right)\Big|_0^\infty\ .
\end{aligned} \tag{13}$$

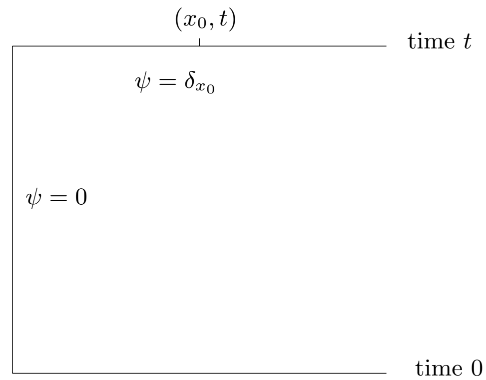

Figure 2: *The boundary and final-time conditions for ψ.*

(I've used here that the heat equation backward-in-time is the formal adjoint of the heat equation forward-in-time; you saw this before in the discussion of the forward Kolmogorov equation, which is always the formal adjoint of the backward Kolmogorov equation.) Because of our special choice of ψ the last formula simplifies: ψ and ψ_y decay rapidly enough at ∞ to kill the "boundary term at infinity," and the fact that $\psi = 0$ at $y = 0$ kills one of the two boundary terms at 0. Since $w(0, s) = \phi(s)$ what remains is

$$\frac{d}{ds} \int_0^\infty \psi(y, s) w(y, s)\, dy = \psi_y(0, s)\phi(s).$$

We're essentially done. Substitution of (12) in the above gives, after integration in s,

$$\int_0^\infty \psi(y, t) w(y, t)\, dy - \int_0^\infty \psi(y, 0) w(y, 0) = \int_0^t G_y(x_0, 0, t - s)\phi(s)\, ds.$$

The first term on the left is just $w(x_0, t)$, by our choice of ψ, and the second term on the left vanishes since $w = 0$ at time 0, yielding precisely the desired solution formula (11).

Final remark: the compatibility condition (10) represents no real loss of generality. If, in the original problem for u, the boundary data have $\phi(0) \neq 0$ then we may simply consider $u - c$ where c is constant. It still solves the heat equation, with boundary data $\phi - c$ and initial data $g - c$. When $c = \phi(0)$ we see that the boundary data vanish at 0. Thus the argument given above applies without difficulty to $u - \phi(0)$.

****

Answering a question left over from Section 2. Remember Pontryagin's maximum principle: it says that for the deterministic control problem with equation of state $dy/ds = f(y, \alpha)$ and value function

$$u(x, t) = \max_\alpha \left\{ \int_t^T h(y(s), \alpha(s))\, ds + g(y(T)) \right\}$$

the optimal path solves the Hamiltonian system

$$\begin{aligned}\frac{dy}{ds} &= \nabla_\pi H(\pi, y) \\ \frac{d\pi}{ds} &= -\nabla_y H(\pi, y)\end{aligned}$$

where $H(\pi, y) = \max_\alpha\{\pi \cdot f(y, \alpha) + h(y, \alpha)\}$ is the Hamiltonian.

I made the further assertion that

$$\pi(s) = \nabla u(y(s), s), \tag{14}$$

evaluated of course along the optimal path $y(s)$. Let us check that $\nabla u(y(s), s)$ does indeed solve the second equation in the Hamiltonian system. (The fact that $y(s)$ solves the first equation was verified in Section 2; this was easy, since $\nabla_\pi H = f$.) The argument works in any dimension, however it is most transparent in 1D so let's work there. Obviously

$$\frac{d}{ds}u_x(y(s), s) = u_{xx}\frac{dy}{ds} + u_x s = u_{xx} f + u_{xs},$$

evaluated as usual at $x = y(s)$. Now consider the Hamilton-Jacobi-Bellman equation

$$u_t(x, t) + \max_\alpha\{u_x(x, t) f(x, \alpha) + h(x, \alpha)\} = 0.$$

Let α_* be the optimal α, and ignore (this is admittedly a formal calculation) the possibility that α_* might not depend smoothly on x and t at some points. Writing the HJB equation as

$$u_t(x, t) + u_x(x, t) f(x, \alpha_*) + h(x, \alpha_*) = 0$$

we differentiate it in x using chain rule. The terms involving derivatives with respect to α_* drop out (because α_* is optimal), so

$$u_{xt}(x, t) + u_{xx}(x, t) f(x, \alpha_*(x, t)) + u_x(x, t) f_x(x, \alpha_*) + h_x(x, \alpha_*) = 0.$$

Making the substitution $x = y(t)$, and remembering that

$$\nabla_x H(\pi, x) = \pi f_x + h_x$$

(evaluated of course at the optimal α_*), we deduce that

$$\frac{d}{dt}u_x(y(t), t) = -(\nabla_x H)(u_x(y(t)), t), y(t))$$

as asserted.

Now consider the mistake in Section 2 which I corrected at the beginning of Section 3. The mistaken assertion was that we always have $\pi(T) = \nabla g(y(T))$ at the final time T. It is tempting to say this, by passing to the limit $t \to T$ in (14). The argument is correct – and the assertion is valid – if $\nabla u(x, t)$ is a continuous function of x and t near $x = y(T)$ and

$t = T$. However this isn't always the case. In fact it fails in Example 1. There we had that $u(x,t) = \phi(t)x^\gamma$ with $\gamma < 1$. Our formula for ϕ has the property that

$$\phi \approx e^{-\rho t}(T-t)^(1-\gamma) \quad \text{near } t = T.$$

Setting $\rho = 0$ for simplicity, we see that

$$u_x(x,t) \approx \gamma x^{\gamma-1}(T-t)^{1-\gamma} = \left(\frac{T-t}{x}\right)^{1-\gamma}$$

near $x = 0$, $t = T$. Therefore the limiting value of $u_x(y(t),t)$ as $t \to T$ need *not* be zero, even though $g = 0$ in this example. Rather, the limit is determined by the slope of $y(t)$ as it approaches 0 at $t = T$.