PDE for Finance, Spring 2003 – Homework 2
Distributed 2/10/03, due 2/24/03.

1) Consider the linear heat equation $u_t - u_{xx} = 0$ in one space dimension, with discontinuous initial data

$$u(x,0) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0. \end{cases}$$

(a) Show by evaluating the solution formula that

$$u(x,t) = N\left(\frac{x}{\sqrt{2t}}\right) \tag{1}$$

where N is the cumulative normal distribution

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-s^2/2}\, ds.$$

(b) Explore the solution by answering the following: what is $\max_x u_x(x,t)$ as a function of time? Where is it achieved? What is $\min_x u_x(x,t)$? For which x is $u_x > (1/10)\max_x u_x$? Sketch the graph of u_x as a function of x at a given time $t > 0$.

(c) Show that $v(x,t) = \int_{-\infty}^{x} u(z,t)\, dz$ solves $v_t - v_{xx} = 0$ with $v(x,0) = \max\{x, 0\}$. Deduce the qualitative behavior of $v(x,t)$ as a function of x for given t: how rapidly does v tend to 0 as $x \to -\infty$? What is the behavior of v as $x \to \infty$? What is the value of $v(0,t)$? Sketch the graph of $v(x,t)$ as a function of x for given $t > 0$.

2) We showed, in the Section 2 notes, that the solution of

$$w_t = w_{xx} \quad \text{for } t > 0 \text{ and } x > 0, \text{ with } w = 0 \text{ at } t = 0 \text{ and } w = \phi \text{ at } x = 0$$

is

$$w(x,t) = \int_0^t \frac{\partial G}{\partial y}(x, 0, t-s)\phi(s)\, ds \tag{2}$$

where $G(x,y,s)$ is the probability that a random walker, starting at x at time 0, reaches y at time s without first hitting the barrier at 0. (Here and throughout this problem set, the random walker solves $dy = \sqrt{2}dw$, i.e. it executes the scaled Brownian whose backward Kolmogorov equation is $u_t + u_{xx} = 0$.) Let's give an alternative demonstration of this fact, following the line of reasoning at the end of the Section 1 notes.

(a) Express, in terms of G, the probability that the random walker (starting at x at time 0) hits the barrier before time t. Differentiate in t to get the probability that it hits the barrier at time t. (This is known as the *first passage time* density).

(b) Use the forward Kolmogorov equation and integration by parts to show that the first passage time density is $\frac{\partial G}{\partial y}(x, 0, t)$.

(c) Deduce the formula (2).

3) Give "solution formulas" for the following initial-boundary-value problems for the linear heat equation

$$w_t - w_{xx} = 0 \quad \text{for } t > 0 \text{ and } x > 0$$

with the specified initial and boundary conditions.

(a) $w_1 = 0$ at $x = 0$; $w_1 = 1$ at $t = 0$. Express your solution in terms of the function $u(x,t)$ defined in Problem 1.

(b) $w_2 = 0$ at $x = 0$; $w_2 = (x - K)_+$ at $t = 0$, with $K > 0$. Express your solution in terms of the function $v(x,t)$ defined in Problem 1(c).

(c) $w_3 = 0$ at $x = 0$; $w_3 = (x - K)_+$ at $t = 0$, with $K < 0$.

(d) $w_4 = 1$ at $x = 0$; $w_4 = 0$ at $t = 0$.

Interpret each as the expected payoff of a suitable barrier-type instrument, whose underlying executes the scaled Brownian motion $dy = \sqrt{2}dw$ with initial condition $y(0) = x$ and an absorbing barrier at 0. (Example: $w_1(x,T)$ is the expected payoff of an instrument which pays 1 at time T if the underlying has not yet hit the barrier and 0 otherwise.)

4) The Section 2 notes reduce the Black-Scholes PDE to the heat equation by brute-force algebraic substitution. This problem achieves the same reduction by a probabilistic route. Our starting point is the fact that

$$V(s,t) = e^{-r(T-t)} E_{y(t)=s} \left[\Phi(y(T)\right] \tag{3}$$

where $dy = rydt + \sigma ydw$.

(a) Consider $z = \frac{1}{\sigma}\log y$. By Ito's formula it satisfies $dz = \frac{1}{\sigma}(r - \frac{1}{2}\sigma^2)dt + dw$. Express the right hand side of (3) as a discounted expected value with respect to z process.

(b) The z process is Brownian motion with drift $\mu = \frac{1}{\sigma}(r - \frac{1}{2}\sigma^2)$. The Cameron-Martin-Girsanov theorem tells how to write an expected value relative to z as a weighted expected value relative to the standard Brownian motion w. Specifically:

$$E_{z(t)=x}\left[F(z(T))\right] = E_{w(t)=x}\left[e^{\mu w(T) - \frac{1}{2}\mu^2(T-t)} F(w(T))\right] \tag{4}$$

where left side is an expectation using the path-space measure associated with z, and the right hand side is an expectation using the path-space measure associated with Brownian motion. Apply this to get an expression for $V(s,t)$ whose right hand side involves an expected value relative to Brownian motion.

(c) An expected payoff relative to Brownian motion is described by the heat equation (more precisely by an equation of the form $u_t + \frac{1}{2}u_{xx} = 0$). Thus (b) expresses the solution of the Black-Scholes PDE in terms of a solution of the heat equation. Verify that this representation is the same as the one given in the Section 2 notes.

5) As noted in Problem 4(b), questions about Brownian motion with drift can often be answered using the Cameron-Martin-Girsanov theorem. But we can also study this process directly. Let's do so now, for the process $dz = \mu dt + dw$ with an absorbing barrier at $z = 0$.

(a) Suppose the process starts at $z_0 > 0$ at time 0. Let $G(z_0, z, t)$ be the probability that the random walker is at position z at time t (and has not yet hit the barrier). Show that

$$G(z_0, z, t) = \frac{1}{\sqrt{2\pi t}} e^{-|z-z_0-\mu t|^2/2t} - \frac{1}{\sqrt{2\pi t}} e^{-2\mu z_0} e^{-|z+z_0-\mu t|^2/2t}.$$

(Hint: just check that this G solves the relevant forward Kolmogorov equation, with the appropriate boundary and initial conditions.)

(b) Show that the first passage time density is

$$\frac{\partial G}{\partial z}(z_0, 0, t) = \frac{2z_0}{t\sqrt{2\pi t}} e^{-|z_0+\mu t|^2/2t}.$$