PDE for Finance, Spring 2003 – Homework 4
Distributed 3/10/03, due 3/31/03.

Problems 1 – 4 concern deterministic optimal control (Section 4 material); problems 5 – 7 concern stochastic control (Section 5 material). Warning: this problem set is longer than usual (mainly because Problems 1 – 4, though not especially difficult, are fairly laborious.)

1) Consider the finite-horizon utility maximization problem with discount rate ρ. The dynamical law is thus

$$dy/ds = f(y(s), \alpha(s)), \quad y(t) = x,$$

and the optimal utility discounted to time 0 is

$$u(x,t) = \max_{\alpha \in A} \left\{ \int_t^T e^{-\rho s} h(y(s), \alpha(s))\, ds + e^{-\rho T} g(y(T)) \right\}.$$

It is often more convenient to consider, instead of u, the optimal utility discounted to time t; this is

$$v(x,t) = e^{\rho t} u(x,t) = \max_{\alpha \in A} \left\{ \int_t^T e^{-\rho(s-t)} h(y(s), \alpha(s))\, ds + e^{-\rho(T-t)} g(y(T)) \right\}.$$

(a) Show (by a heuristic argument similar to those in the Section 4 notes) that v satisfies

$$v_t - \rho v + H(x, \nabla v) = 0$$

with Hamiltonian

$$H(x,p) = \max_{a \in A} \{ f(x,a) \cdot p + h(x,a) \}$$

and final-time data

$$v(x,T) = g(x).$$

(Notice that the PDE for v is autonomous, i.e. there is no explicit dependence on time.)

(b) Now consider the analogous infinite-horizon problem, with the same equation of state, and value function

$$\bar{v}(x,t) = \max_{\alpha \in A} \int_t^\infty e^{-\rho(s-t)} h(y(s), \alpha(s))\, ds.$$

Show (by an elementary comparison argument) that $\bar{v}$ is independent of t, i.e. $\bar{v} = \bar{v}(x)$ is a function of x alone. Conclude using part (a) that if $\bar{v}$ is finite, it solves the stationary PDE

$$-\rho \bar{v} + H(x, \nabla \bar{v}) = 0.$$

2) Recall Example 1 of the Section 4 notes: the state equation is $dy/ds = ry - \alpha$ with $y(t) = x$, and the value function is

$$u(x,t) = \max_{\alpha \geq 0} \int_t^\tau e^{-\rho s} h(\alpha(s))\, ds$$

with $h(a) = a^\gamma$ for some $0 < \gamma < 1$, and

$$\tau = \begin{cases} \text{first time when } y = 0 \text{ if this occurs before time } T \\ T \text{ otherwise.} \end{cases}$$

(a) We obtained a formula for $u(x,t)$ in the Section 4 notes, however our formula doesn't make sense when $\rho - r\gamma = 0$. Find the correct formula in that case.

(b) Let's examine the infinite-horizon-limit $T \to \infty$. Following the lead of Problem 1 let us concentrate on $v(x,t) = e^{\rho t} u(x,t)$ = optimal utility discounted to time t. Show that

$$\bar{v}(x) = \lim_{T \to \infty} v(x,t) = \begin{cases} G_\infty x^\gamma & \text{if } \rho - r\gamma > 0 \\ \infty & \text{if } \rho - r\gamma \leq 0 \end{cases}$$

with $G_\infty = [(1-\gamma)/(\rho - r\gamma)]^{1-\gamma}$.

(c) Use the stationary PDE of Problem 1(b) (specialized to this example) to obtain the same result.

(d) What is the optimal consumption strategy, for the infinite-horizon version of this problem?

3) Consider the analogue of Example 1 with the power-law utility replaced by the logarithm: $h(a) = \ln a$. To avoid confusion let us write u_γ for the value function obtained in the notes using $h(a) = a^\gamma$, and $u_{\log}$ for the value function obtained using $h(a) = \ln a$. Recall that $u_\gamma(x,t) = g_\gamma(t) x^\gamma$ with

$$g_\gamma(t) = e^{-\rho t} \left[\frac{1-\gamma}{\rho - r\gamma} \left(1 - e^{-\frac{(\rho - r\gamma)(T-t)}{1-\gamma}} \right) \right]^{1-\gamma}.$$

(a) Show, by a direct comparison argument, that

$$u_{\log}(\lambda x, t) = u_{\log}(x,t) + \frac{1}{\rho} e^{-\rho t} (1 - e^{-\rho(T-t)}) \ln \lambda$$

for any $\lambda > 0$. Use this to conclude that

$$u_{\log}(x,t) = g_0(t) \ln x + g_1(t)$$

where $g_0(t) = \frac{1}{\rho} e^{-\rho t}(1 - e^{-\rho(T-t)})$ and g_1 is an as-yet unspecified function of t alone.

(b) Pursue the following scheme for finding g_1: Consider the utility $h = \frac{1}{\gamma}(a^\gamma - 1)$. Express its value function u_h in terms of u_γ. Now take the limit $\gamma \to 0$. Show this gives a result of the expected form, with

$$g_0(t) = g_\gamma(t)|_{\gamma=0}$$

and

$$g_1(t) = \left.\frac{dg_\gamma}{d\gamma}(t)\right|_{\gamma=0}.$$

(This leads to an explicit formula for g_1 but it's messy; I'm not asking you to write it down.)

(c) Indicate how g_0 and g_1 could alternatively have been found by solving appropriate PDE's. (Hint: find the HJB equation associated with $h(a) = \ln a$, and show that the ansatz $u_{\log} = g_0(t) \ln x + g_1(t)$ leads to differential equations that determine g_0 and g_1.)

4) Our Example 1 considers an investor who receives interest (at constant rate r) but no wages. Let's consider what happens if the investor also receives wages at constant rate w. The equation of state becomes

$$dy/ds = ry + w - \alpha \quad \text{with } y(t) = x,$$

and the value function is

$$u(x,t) = \max_{\alpha \geq 0} \int_t^T e^{-\rho s} h(\alpha(s))\, ds$$

with $h(a) = a^\gamma$ for some $0 < \gamma < 1$. Since the investor earns wages, we now permit $y(s) < 0$, however we insist that the final-time wealth be nonnegative ($y(T) \geq 0$).

(a) Which pairs (x,t) are acceptable? The strategy that maximizes $y(T)$ is clearly to consume nothing ($\alpha(s) = 0$ for all $t < s < T$). Show this results in $y(T) \geq 0$ exactly if

$$x + \phi(t)w \geq 0$$

where

$$\phi(t) = \frac{1}{r}\left(1 - e^{-r(T-t)}\right).$$

Notice for future reference that ϕ solves $\phi' - r\phi + 1 = 0$ with $\phi(T) = 0$.

(b) Find the HJB equation that $u(x,t)$ should satisfy in its natural domain $\{(x,t) : x + \phi(t)w \geq 0\}$. Specify the boundary conditions when $t = T$ and where $x + \phi w = 0$.

(c) Substitute into this HJB equation the ansatz

$$v(x,t) = e^{-\rho t} G(t)(x + \phi(t)w)^\gamma.$$

Show v is a solution when G solves the familiar equation

$$G_t + (r\gamma - \rho)G + (1-\gamma)G^{\gamma/(\gamma-1)} = 0$$

(the same equation we solved in Example 1). Deduce a formula for v.

(d) In view of (a), a more careful definition of the value function for this control problem is

$$u(x,t) = \max_{\alpha \geq 0} \int_t^{\tau} e^{-\rho s} h(\alpha(s))\, ds$$

where

$$\tau = \begin{cases} \text{first time when } y(s) + \phi(s)w = 0 \text{ if this occurs before time } T \\ T \text{ otherwise.} \end{cases}$$

Use a verification argument to prove that the function v obtained in (c) is indeed the value function u defined this way.

5) Our geometric Example 2 gave $|\nabla u| = 1$ in D (with $u = 0$ at ∂D) as the HJB equation associated with starting at a point x in some domain D, traveling with speed at most 1, and arriving at ∂D as quickly as possible. Let's consider what becomes of this problem when we introduce a little noise. The state equation becomes

$$dy = \alpha(s)ds + \epsilon dw, \quad y(0) = x,$$

where $\alpha(s)$ is a (non-anticipating) control satisfying $|\alpha(s)| \leq 1$, y takes values in R^n, and each component of w is an independent Brownian motion. Let $\tau_{x,\alpha}$ denote the arrival time:

$$\tau_{x,\alpha} = \text{time when } y(s) \text{ first hits } \partial D,$$

which is of course random. The goal is now to minimize the *expected* arrival time at ∂D, so the value function is

$$u(x) = \min_{|\alpha(s)| \leq 1} E_{y(0)=x} \left\{ \tau_{x,\alpha} \right\}.$$

(a) Show, using an argument similar to that in the Section 5 notes, that u solves the PDE

$$1 - |\nabla u| + \tfrac{1}{2}\epsilon^2 \Delta u = 0 \quad \text{in } D$$

with boundary condition $u = 0$ at ∂D.

(b) Your answer to (a) should suggest a specific feedback strategy for determining $\alpha(s)$ in terms of $y(s)$. What is it?

6) Let's solve the differential equation from the last problem explicitly, for the special case when $D = [-1, 1]$:

$$\begin{aligned} 1 - |u_x| + \tfrac{1}{2}\epsilon^2 u_{xx} &= 0 \quad \text{for } -1 < x < 1 \\ u &= 0 \quad \text{at } x = \pm 1. \end{aligned}$$

(a) Assuming that the solution u is unique, show it satisfies $u(x) = u(-x)$. Conclude that $u_x = 0$ and $u_{xx} < 0$ at $x = 0$. Thus u has a maximum at $x = 0$.

(b) Notice that $v = u_x$ solves $1 - |v| + \delta v_x = 0$ with $\delta = \frac{1}{2}\epsilon^2$. Show that

$$\begin{aligned} v &= -1 + e^{-x/\delta} \qquad \text{for } 0 < x < 1 \\ v &= +1 - e^{x/\delta} \qquad \text{for } -1 < x < 0. \end{aligned}$$

Integrate once to find a formula for u.

(c) Verify that as $\epsilon \to 0$, this solution approaches $1 - |x|$.

[Comment: the assumption of uniqueness in part (a) is convenient, but it can be avoided. Outline of how to do this: observe that any critical point of u must be a local maximum (since $u_x = 0$ implies $u_{xx} < 0$). Therefore u has just one critical point, say x_0, which is a maximum. Get a formula for u by arguing as in (b). Then use the boundary condition to see that x_0 had to be 0.]

7) Let's consider what becomes of Merton's optimal investment and consumption problem if there are two risky assets: one whose price satisfies $dp_2 = \mu_2 p_2 dt + \sigma_2 p_2 dw_2$ and another whose price satisfies $dp_3 = \mu_3 p_3 dt + \sigma_3 p_3 dw_3$. To keep things simple let's suppose w_2 and w_3 are independent Brownian motions. It is natural to assume $\mu_2 > r$ and $\mu_3 > r$ where r is the risk-free rate. (Why?) Let $\alpha_2(s)$ and $\alpha_3(s)$ be the proportions of the investor's total wealth invested in the risky assets at time s, so that $1 - \alpha_2 - \alpha_3$ is the proportion of wealth invested risk-free. Let β be the rate of consumption. Then the investor's wealth satisfies

$$dy = (1 - \alpha_2 - \alpha_3) y r ds + \alpha_2 y(\mu_2 ds + \sigma_2 dw_2) + \alpha_3 y(\mu_3 ds + \sigma_3 dw_3) - \beta ds.$$

(Be sure you understand this; but you need not explain it on your solution sheet.) Use the power-law utility: the value function is thus

$$u(x,t) = \max_{\alpha_2, \alpha_3, \beta} E_{y(t)=x} \left[\int_t^\tau e^{-\rho s} \beta^\gamma(s)\, ds \right]$$

where τ is the first time $y(s) = 0$ if this occurs, or $\tau = T$ otherwise.

(a) Derive the HJB equation.

(b) What is the optimal investment policy (the optimal choice of α_2 and α_3)? What restriction do you need on the parameters to be sure $\alpha_2 > 0$, $\alpha_3 > 0$, and $\alpha_2 + \alpha_3 < 1$?

(c) Find a formula for $u(x,t)$. [Hint: the nonlinear equation you have to solve is not really different from the one considered in Section 5.]