PDE for Finance Notes, Spring 2003 – Sample Exam Topics
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Below, section-by-section, are some types of questions that might be on the final exam. This list is exemplary, not comprehensive. At the end are a few actual questions from the Spring 1999 PDE for Finance final.

- Section 1 material:
  - Consider the stochastic differential equation $dy = f ds + g dw$ with infinitesimal generator $\mathcal{L}u = fu_x + (1/2)g^2 u_{xx}$. Show that if $\phi(x, t)$ satisfies $\phi_t + \mathcal{L}\phi \leq 0$ then $E_{y(t)=x}[\phi(y(T), T)] \leq \phi(x, t)$ for every $T > t$.
  - Consider a random walker executing scaled Brownian motion with drift, $dy = \mu dt + \sigma dw$, starting at $0 < x < 1$ at time 0. What PDE describes the probability that its first exit from $[0, 1]$ is at 1? Evaluate this probability by solving the PDE. (Hint: look for a solution of the form $ae^{\alpha x} + b$.)
  - Consider the pde ..... For what stochastic process is it the forward Kolmogorov equation? What is the probabilistic interpretation of its solution? Is this PDE the backward Kolmogorov equation of any stochastic process?
- Section 2 material:
  - Consider a random walker executing $dy = f dt + g dw$. Let $G(x, y, t)$ be the probability of being at y at time t, if it starts at x at time 0. When is G symmetric in x and y, i.e. when does $G(x, y, t) = G(y, x, t)$? Explain.
  - Consider a random walker executing scaled Brownian motion $dy = \sqrt{2}dw$ starting from $x > B$ at time 0. Consider a "knockout call with barrier B and payoff g;" this option pays $g(y(T))$ if the walker survives to maturity T without crossing $x = B$, and 0 otherwise. Let $u(x)$ be the expected payoff. What initial-boundary-value problem can we solve to evaluate u? There there is a solution formula in the form $u(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi T}} e^{(x-y)^2/4T} \phi(y)\, dy$. What is the function $\phi(y)$?
  - Let $v(x, t)$ solve the linear heat equation $u_t = u_{xx}$ with initial condition $v(x, 0) = \max x, 0$, and let $u(x, t) = \partial v/\partial x$. Express, in terms of u and v, the solutions of the following initial-boundary-value problems ... (like HW2 Problem 3).
- Section 3 material:
  - Suppose $u_t - u_{xx} = 0$ for $x \in R$ and $t > 0$, with $u = u_0(x)$ at $t = 0$ and reasonable growth as $|x| \to \infty$. Suppose furthermore that u_0 is a convex function of x, i.e. $u_{0xx} \geq 0$. Show that $u_{xx} \geq 0$ for all x and all $t > 0$. Deduce that $u_t \geq 0$ for all x and all $t > 0$.
  - Suppose $u_t - u_{xx} + a(x, t)u = 0$ for $0 < x < 1$, with $u(0, t) = u(1, t) = 0$ for all $t > 0$. Suppose in addition $u > 0$ at $t = 0$. Show that $u > 0$ for all t.
  - Consider the simple finite-difference scheme ... for solving $u_t = u_{xx}$. Why is it important that $\Delta t < (1/2)(\Delta x)^2$?

- Section 4 material:
  - Consider the following optimal control problem.... Identify an appropriate value function, and specify the Hamilton-Jacobi equation it solves. Assuming you know the value function, describe an optimal control strategy.
- Section 5 material:
  - Consider the Merton portfolio problem with utility c^γ (statement of problem here). Show (without finding or using the explicit solution) that the value function satisfies $u(ax, t) = a^\gamma u(x, t)$ for any a.
  - Explain the following statement: "Perturbing a deterministic optimal control problem by introducing a little noise in the dynamics corresponds to adding a small diffusion term to the HJB equation."
- Section 6 material:
  - Consider the perpetual American put. The underlying solves $dy = \mu y ds + \sigma y dw$ and the goal is to evaluate $\max_\tau E_{y(0)=x}\left[e^{-r\tau}(K - y(\tau))_+\right]$. Show that if v is a differentiable function with $v \geq (K - x)_+$ and $-rv + \mu x v_x + \frac{1}{2}\sigma^2 x^2 v_{xx} \leq 0$ for all x then v gives an upper bound: $E_{y(0)=x}\left[e^{-r\tau}(K - y(\tau))_+\right] \leq v(x)$ for any bounded, nonanticipating stopping time τ.
  - You hold a perpetual option with payoff y^2. The underlying has Ornstein-Uhlenbeck dynamics $dy = -y ds + dw$. Your goal is to maximize your discounted expected payoff: $\max_\tau E_{y(0)=x}\left[e^{-r\tau} y^2(\tau)\right]$. Consider taking $\tau = \tau_h$ to be the first time $|y| = h$. What PDE describes the discounted expected payoff achieved by τ_h?
- Section 7 material:
  - Consider the following discrete-time stochastic dynamic programming problem.... Let $J_i(w_i)$ be the optimal expected value starting at time i, if the state at time i is w_i. What relation does the dynamic programming principle give establish between J_i and J_{i-1}?
- Section 8 material:
  - Consider a perpetual American put on an underlying with jump-diffusion dynamics $dy = \mu y dt + \sigma y dw + J dN$. Consider, as usual, a stopping time of the form τ_h = first time $y(t)$ arrives at h. What boundary-value problem describes the expected discounted payoff achieved by τ_h?

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Some sample exam questions. Here are two sample questions. They address just a few of the topics we've covered. They should however give you a rough idea what to expect, in terms of depth and level of difficulty.

1. Consider the stochastic control problem with state equation

$$dy = f(y,\alpha)ds + g(y,\alpha)dw, \quad y(t) = x.$$

and value function

$$u(x,t) = \max_{\alpha} E_{y(t)=x}\left[g(y(T))\right].$$

(We assume y is scalar-valued for simplicity.) The associated Hamilton-Jacobi-Bellman equation is

$$u_t + \max_a [f(x,a)u_x + \tfrac{1}{2}g^2(x,a)u_{xx}] = 0 \quad \text{for } t < T$$

with final-time condition $u = g$ at $t = T$.

(a) Show that if $v(x,t)$ is a smooth solution of this final-value problem, and if α is any nonanticipating control, then $E_{y(t)=x}\left[g(y(T))\right] \le v(x,t)$.

(b) Why is the assertion of (a) called a "verification theorem"?

2. Consider the lognormal random walk with constant drift and volatility

$$dy = \mu y ds + \sigma y dw$$

with $\mu \neq \frac{1}{2}\sigma^2$. Let $\tau(x)$ be the first exit time from the interval $[a,b]$, when the initial position is $y(0) = x$, with $0 < a < x < b < \infty$. You may use, if convenient, the following facts:

- The process does exit from $[a,b]$ with prob. 1, and the mean exit time is finite.
- The function $u(x) = \log x$ satisfies $\mu x u_x + \frac{1}{2}\sigma^2 x^2 u_{xx} = \mu - \frac{1}{2}\sigma^2$.
- The function $u(x) = x^\gamma$ satisfies $\mu x u_x + \frac{1}{2}\sigma^2 x^2 u_{xx} = 0$ when $\gamma = 1 - (2\mu/\sigma^2)$.

(a) Let p_a (respectively $p_b = 1 - p_a$) be the probability that the first exit is at a (respectively b). Show that

$$p_a = \frac{b^\gamma - x^\gamma}{b^\gamma - a^\gamma}, \quad p_b = \frac{x^\gamma - a^\gamma}{b^\gamma - a^\gamma}.$$

[Hint: apply Ito's lemma to $\phi(y) = y^\gamma$.]

(b) Show that the expected exit time $E_{y(0)=x}\left[\tau\right]$ is equal to

$$(\mu - \tfrac{1}{2}\sigma^2)^{-1}(p_a \log a + p_b \log b - \log x).$$

[Hint: apply Ito's lemma to $\phi(y) = \log y$.]