

## PDE for Finance, Spring 2014 – Homework 1

Distributed 1/27/2014, due 2/10/2014. HW must be turned in by the due date to get credit, unless an extension has been granted.

1) Consider the lognormal random walk

$$dy = \mu y dt + \sigma y dw$$

starting at  $y(0) = x$ . Assume  $\mu \neq \frac{1}{2}\sigma^2$ . The Section 1 notes examine the mean exit time from an interval  $[a, b]$  where  $0 < a < x < b$ . There we used the PDE for the mean exit time

$$\mu x u_x + \frac{1}{2}\sigma^2 x^2 u_{xx} = -1 \quad \text{for } a < x < b \quad (1)$$

with boundary conditions  $u(a) = u(b) = 0$  to derive an explicit formula for  $u$ .

(a) Show that the general solution of (1), without taking any boundary conditions into account, is

$$u = \frac{1}{\frac{1}{2}\sigma^2 - \mu} \log x + c_1 + c_2 x^\gamma$$

with  $\gamma = 1 - 2\mu/\sigma^2$ . Here  $c_1$  and  $c_2$  are arbitrary constants. [The formula given in the notes for the mean exit time is easy to deduce from this fact, by using the boundary conditions to solve for  $c_1$  and  $c_2$ ; however you need not do this calculation as part of your homework.]

(b) Argue as in the notes to show that the mean exit time from the interval  $(a, b)$  is finite. (Hint: mimic the argument used to answer Question 3, using  $\phi(y) = \log y$ .)

(c) Let  $p_a$  be the probability that the process exits at  $a$ , and  $p_b = 1 - p_a$  the probability that it exits at  $b$ . Give an equation for  $p_a$  in terms of the barriers  $a, b$  and the initial value  $x$ . (Hint: mimic the argument used in the answer to Question 4, using  $\phi(y) = y^\gamma$ .) How does  $p_a$  behave in the limit  $a \rightarrow 0$ ?

2) Examine the analogues of Problem 1(a)–(c) when  $\mu = \frac{1}{2}\sigma^2$ . (Hint: notice that  $xu_x + x^2u_{xx} = u_{zz}$  with  $z = \log x$ .)

3) Consider a diffusion  $dy = f(y)ds + g(y)dw$  starting at  $x$  at time 0, with  $a < x < b$ . Let  $\tau$  be its exit time from the interval  $[a, b]$ , and assume  $E[\tau] < \infty$ .

(a) Let  $u_a(x)$  be the probability it exits at  $a$ . Show that  $u_a$  solves  $f u_x + \frac{1}{2}g^2 u_{xx} = 0$  with boundary conditions  $u_a(a) = 1, u_a(b) = 0$ .

(b) Apply this method to Problem 1(c). Is this approach fundamentally different from the one indicated by the hint above?

4) Consider once again a diffusion  $dy = f(y)ds + g(y)dw$  starting at  $x$  at time 0. We know the mean arrival time to the boundary  $v(x) = E[\tau]$  satisfies  $fv_x + \frac{1}{2}g^2v_{xx} = -1$  with  $v = 0$  at  $x = a, b$ . Now consider the *second moment* of the arrival time  $h(x) = E[\tau^2]$ . Show that it satisfies  $fh_x + \frac{1}{2}g^2h_{xx} = -2v(x)$ , with  $h = 0$  at  $x = a, b$ .

5) Let  $w(t)$  be standard Brownian motion, starting at 0. Let  $\tau_n$  be the first time  $w$  exits from the interval  $[-n, 1]$ , and let  $\tau_\infty$  the the first time it reaches  $w = 1$ .

- (a) Find the expected value of  $\tau_n$ , and the probability that the path exits  $[-n, 1]$  at  $-n$ .
- (b) Verify by direct evaluation that  $w(\tau_n)$  has mean value 0. (This must of course be true, since  $E[\int_0^{\tau_n} dw] = 0$  by Dynkin's theorem.)
- (c) Conclude from (a) that  $E[\tau_\infty] = \infty$ .
- (d) Show that  $\tau_\infty$  is almost-surely finite.