

**PDE for Finance, Spring 2014 – Homework 2**

**Distributed 2/11/14, due 2/24/14.**

*Typo in 1(b) corrected 2/17: a factor of  $\sigma^2$  was missing before.*

*Typo in 4(a) corrected 2/18: I wrote  $t \rightarrow 0$  before, but I meant  $t \rightarrow \infty$ .*

- (1) This problem uses a PDE to value a zero-coupon bond when the short-term interest rate is described by the Vasicek model. Suppose  $r(t)$  solves  $dr = (\theta - ar)dt + \sigma dw$ , where  $\theta$ ,  $a$ , and  $\sigma$  are positive constants. If today is time  $t_0$  and the short-term rate today is  $r(t_0) = r_0$ , the value of a zero-coupon bond with maturity  $T$  and face value of one dollar is

$$E_{r(t_0)=r_0} \left[ e^{-\int_{t_0}^T r(s) ds} \right].$$

- (a) Explain why this is equal to  $V(t_0, r(t_0))$ , where  $V(t, r)$  solves the PDE

$$V_t + (\theta - ar)V_r + \frac{1}{2}\sigma^2 V_{rr} - rV = 0$$

for  $t < T$ , with the final-time condition  $V(T, r) = 1$  for all  $r$ .

- (b) Look for a solution of the form  $V(t, r) = A(t, T)e^{-B(t, T)r}$ . Show that  $A$  and  $B$  should satisfy

$$A_t - \theta AB + \frac{1}{2}\sigma^2 AB^2 = 0 \quad \text{and} \quad B_t - aB + 1 = 0$$

with final-time conditions

$$A(T, T) = 1 \quad \text{and} \quad B(T, T) = 0.$$

- (c) Solving for  $B$  first, then  $A$ , show that the solution is

$$B(t, T) = \frac{1}{a}(1 - e^{-a(T-t)})$$

and

$$A(t, T) = \exp \left[ \left( \frac{\theta}{a} - \frac{\sigma^2}{2a^2} \right) (B(t, T) - T + t) - \frac{\sigma^2}{4a} B^2(t, T) \right].$$

- (2) Consider the linear heat equation  $u_t - u_{xx} = 0$  in one space dimension, with discontinuous initial data

$$u(x, 0) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0. \end{cases}$$

- (a) Show by evaluating the solution formula that

$$u(x, t) = N \left( \frac{x}{\sqrt{2t}} \right) \tag{1}$$

where  $N$  is the cumulative normal distribution

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-s^2/2} ds.$$

- (b) Explore the solution by answering the following: what is  $\max_x u_x(x, t)$  as a function of time? Where is it achieved? What is  $\min_x u_x(x, t)$ ? For which  $x$  is  $u_x > (1/10) \max_x u_x$ ? Sketch the graph of  $u_x$  as a function of  $x$  at a given time  $t > 0$ .
- (c) Show that  $v(x, t) = \int_{-\infty}^x u(z, t) dz$  solves  $v_t - v_{xx} = 0$  with  $v(x, 0) = \max\{x, 0\}$ . Deduce the qualitative behavior of  $v(x, t)$  as a function of  $x$  for given  $t$ : how rapidly does  $v$  tend to 0 as  $x \rightarrow -\infty$ ? What is the behavior of  $v$  as  $x \rightarrow \infty$ ? What is the value of  $v(0, t)$ ? Sketch the graph of  $v(x, t)$  as a function of  $x$  for given  $t > 0$ .
- (3) This problem obtains convenient representations for the solutions of some particular initial-boundary-value problems for the linear heat equation on the half-line:

$$w_t - w_{xx} = 0 \quad \text{for } t > 0 \text{ and } x > 0.$$

- (a) Let  $w_1$  be the solution with  $w_1 = 0$  at  $x = 0$  and  $w_1 = 1$  at  $t = 0$ . Express it in terms of the function  $u(x, t)$  defined in Problem 2.
- (b) Let  $w_2$  be the solution with  $w_2 = 0$  at  $x = 0$  and  $w_2 = (x - K)_+$  at  $t = 0$ . Assume that  $K > 0$ . Express  $w_2$  in terms of the function  $v(x, t)$  defined in Problem 2(c).
- (c) Let  $w_3$  be the solution with  $w_3 = 0$  at  $x = 0$  and  $w_3 = (x - K)_+$  at  $t = 0$ , when  $K < 0$ . Find a convenient representation of  $w_3$  analogous to those you gave for  $w_1$  and  $w_2$ .
- (d) Let  $w_4$  be the solution with  $w_4 = 1$  at  $x = 0$  and  $w_4 = 0$  at  $t = 0$ . Find a convenient representation, analogous to those you gave for the other  $w_i$ . (Hint: what boundary value problem does  $w_4 - 1$  solve?)
- (e) Interpret each  $w_i$  as the expected payoff of a suitable barrier-type instrument, whose underlying executes the scaled Brownian motion  $dy = \sqrt{2}dw$  with initial condition  $y(0) = x$  and an absorbing barrier at 0. (Example:  $w_1(x, T)$  is the expected payoff of an instrument which pays 1 at time  $T$  if the underlying has not yet hit the barrier and 0 otherwise.)

NOTE: One can, of course, use the general representation formula for solutions of the half-space problem to get a “formula” for each  $w_i$ . But I’m not asking you to do this. Rather, I’m asking you to find (using the functions introduced in Problem 2) a solution of the PDE with the correct initial and boundary conditions. This is *much* easier.

- (4) Let’s look more closely at the function  $w_1$  introduced in Problem 3(a).
- (a) Show that for fixed  $x > 0$ ,  $w_1(x, t) \rightarrow 0$  as  $t \rightarrow \infty$ .
- (b) How fast does it decay? (Suggestion: show that as  $t \rightarrow \infty$ ,  $w_1(x, t) \sim Ct^{-\alpha}$ . What is the best possible value of  $\alpha$ ?)
- (5) The Section 2 notes reduce the Black-Scholes PDE to the heat equation by brute-force algebraic substitution. This problem achieves the same reduction by a probabilistic route. Our starting point is the fact that

$$V(s, t) = e^{-r(T-t)} E_{y(t)=s} [\Phi(y(T))] \tag{2}$$

where  $dy = rydt + \sigma ydw$ .

- (a) Consider  $z = \frac{1}{\sigma} \log y$ . By Ito's formula it satisfies  $dz = \frac{1}{\sigma}(r - \frac{1}{2}\sigma^2)dt + dw$ . Express the right hand side of (2) as a discounted expected value with respect to  $z$  process.
- (b) The  $z$  process is Brownian motion with drift  $\mu = \frac{1}{\sigma}(r - \frac{1}{2}\sigma^2)$ . The Cameron-Martin-Girsanov theorem tells how to write an expected value relative to  $z$  as a weighted expected value relative to the standard Brownian motion  $w$ . Specifically:

$$E_{z(t)=\frac{1}{\sigma} \log s} \left[ \Phi(e^{\sigma z(T)}) \right] = E_{w(t)=\frac{1}{\sigma} \log s} \left[ e^{\mu(w(T)-w(t)) - \frac{1}{2}\mu^2(T-t)} \Phi(e^{\sigma w(T)}) \right] \quad (3)$$

where the left side is an expectation using the path-space measure associated with  $z$ , and the right hand side is an expectation using the path-space measure associated with Brownian motion. Apply this to get an expression for  $V(s, t)$  whose right hand side involves an expected value relative to Brownian motion.

- (c) An expected payoff relative to Brownian motion is described by the heat equation (more precisely by an equation of the form  $u_t + \frac{1}{2}u_{xx} = 0$ ). Thus (b) expresses the solution of the Black-Scholes PDE in terms of a solution of the heat equation. Verify that this representation is the same as the one given in the Section 2 notes.