**PDE for Finance, Spring 2014 – Homework 3. Distributed 3/3/14, due 3/24/14.**

1. We showed, in the Section 2 notes, that the solution of

$$w_t = w_{xx} \quad \text{for } t > 0 \text{ and } x > 0, \text{ with } w = 0 \text{ at } t = 0 \text{ and } w = \phi \text{ at } x = 0$$

is

$$w(x,t) = \int_0^t \frac{\partial G}{\partial y}(x, 0, t-s)\phi(s)\, ds \tag{1}$$

where $G(x, y, s)$ is the probability that a random walker, starting at $x$ at time 0, reaches $y$ at time $s$ without first hitting the barrier at 0. (Here the random walker solves $dy = \sqrt{2}dw$, i.e. it executes the scaled Brownian whose backward Kolmogorov equation is $u_t + u_{xx} = 0$.) Let's give an alternative demonstration of this fact, following the line of reasoning at the end of the Section 1 notes.

   (a) Express, in terms of $G$, the probability that the random walker (starting at $x$ at time 0) hits the barrier before time $t$. Differentiate in $t$ to get the probability that it hits the barrier at time $t$. (This is known as the *first passage time* density).

   (b) Use the forward Kolmogorov equation and integration by parts to show that the first passage time density is $\frac{\partial G}{\partial y}(x, 0, t)$.

   (c) Deduce the formula (1).

2. As noted in HW2 problem 5, questions about Brownian motion with drift can often be answered using the Cameron-Martin-Girsanov theorem. But we can also study this process directly. Let's do so now, for the process $dz = \mu dt + dw$ with an absorbing barrier at $z = 0$.

   (a) Suppose the process starts at $z_0 > 0$ at time 0. Let $G(z_0, z, t)$ be the probability that the random walker is at position $z$ at time $t$ (and has not yet hit the barrier). Show that

$$G(z_0, z, t) = \frac{1}{\sqrt{2\pi t}} e^{-|z - z_0 - \mu t|^2/2t} - \frac{1}{\sqrt{2\pi t}} e^{-2\mu z_0} e^{-|z + z_0 - \mu t|^2/2t}.$$

   (Hint: just check that this $G$ solves the relevant forward Kolmogorov equation, with the appropriate boundary and initial conditions.)

   (b) Show that the first passage time density is

$$\frac{1}{2}\frac{\partial G}{\partial z}(z_0, 0, t) = \frac{z_0}{t\sqrt{2\pi t}} e^{-|z_0 + \mu t|^2/2t}.$$

3. Consider the linear heat equation $u_t - u_{xx} = 0$ on the interval $0 < x < 1$, with boundary condition $u = 0$ at $x = 0, 1$ and initial condition $u = 1$.

   (a) Interpret $u$ as the value of a suitable double-barrier option.

(b) Express $u(t, x)$ as a Fourier sine series, as explained in Section 3.

(c) At time $t = 1/100$, how many terms of the series are required to give $u(t, x)$ within one percent accuracy?

4. Consider the SDE $dy = f(y)dt + g(y)dw$. Let $G(x, y, t)$ be the fundamental solution of the forward Kolmogorov PDE, i.e. the probability that a walker starting at $x$ at time 0 is at $y$ at time $t$. Show that if the infinitesimal generator is self-adjoint, i.e.

$$-(fu)_x + \tfrac{1}{2}(g^2 u)_{xx} = f u_x + \tfrac{1}{2} g^2 u_{xx},$$

then the fundamental solution is symmetric, i.e. $G(x, y, t) = G(y, x, t)$.

5. Consider the stochastic differential equation $dy = f(y, s)ds + g(y, s)dw$, and the associated backward and forward Kolmogorov equations

$$u_t + f(x, t)u_x + \tfrac{1}{2} g^2(x, t)u_{xx} = 0 \quad \text{for } t < T, \text{ with } u = \Phi \text{ at } t = T$$

and

$$\rho_s + (f(z, s)\rho)_z - \tfrac{1}{2}(g^2(z, s)\rho)_{zz} = 0 \quad \text{for } s > 0, \text{ with } \rho(z) = \rho_0(z) \text{ at } s = 0.$$

Recall that $u(x, t)$ is the expected value (starting from $x$ at time $t$) of payoff $\Phi(y(T))$, whereas $\rho(z, s)$ is the probability distribution of the diffusing state $y(s)$ (if the initial distribution is $\rho_0$).

(a) The solution of the backward equation has the following property: if $m = \min_z \Phi(z)$ and $M = \max_z \Phi(z)$ then $m \le u(x, t) \le M$ for all $t < T$. Give two distinct justifications: one using the maximum principle for the PDE, the other using the probabilistic interpretation.

(b) The solution of the forward equation does *not* in general have the same property; in particular, $\max_z \rho(z, s)$ can be larger than the maximum of $\rho_0$. Explain why not, by considering the example $dy = -yds$. (Intuition: $y(s)$ moves toward the origin; in fact, $y(s) = e^{-s}y_0$. Viewing $y(s)$ as the position of a moving particle, we see that particles tend to collect at the origin no matter where they start. So $\rho(z, s)$ should be increasingly concentrated at $z = 0$.) Show that the solution in this case is $\rho(z, s) = e^s \rho_0(e^s z)$. This counterexample has $g = 0$; can you also give a counterexample using $dy = -yds + \epsilon dw$?

6. The solution of the forward Kolmogorov equation is a probability density, so we expect it to be nonnegative (assuming the initial condition $\rho_0(z)$ is everywhere nonnegative). In light of Problem 2b it's natural to worry whether the PDE has this property. Let's show that it does.

(a) Consider the initial-boundary-value problem

$$w_t = a(x, t)w_{xx} + b(x, t)w_x + c(x, t)w$$

with $x$ in the interval $(0, 1)$ and $0 < t < T$. We assume as usual that $a(x, t) > 0$. Suppose furthermore that $c < 0$ for all $x$ and $t$. Show that if $0 \le w \le M$ at the initial time and the spatial boundary then $0 \le w \le M$ for all $x$ and $t$. (Hint: a positive maximum cannot be achieved in the interior or at the final boundary. Neither can a negative minimum.)

(b) Now consider the same PDE but with $\max_{x,t} c(x,t)$ positive. Suppose the initial and boundary data are nonnegative. Show that the solution $w$ is nonnegative for all $x$ and $t$. (Hint: apply part (a) not to $w$ but rather to $\bar{w} = e^{-Ct} w$ with a suitable choice of $C$.)

(c) Consider the solution of the forward Kolmogorov equation in the interval, with $\rho = 0$ at the boundary. (It represents the probability of arriving at $z$ at time $s$ without hitting the boundary first.) Show using part (b) that $\rho(z,s) \ge 0$ for all $s$ and $z$.

[Comment: statements analogous to (a)-(c) are valid for the initial-value problem as well, when we solve for all $x \in R$ rather than for $x$ in a bounded domain. The justification takes a little extra work however, and it requires some hypothesis on the growth of the solution at $\infty$.]

7. Consider the solution of

$$u_t + a u_{xx} = 0 \quad \text{for } t < T, \text{ with } u = \Phi \text{ at } t = T$$

where $a$ is a positive constant. Recall that in the stochastic interpretation, $a$ is $\frac{1}{2} g^2$ where $g$ represents volatility. Let's use the maximum principle to understand qualitatively how the solution depends on volatility.

(a) Show that if $\Phi_{xx} \ge 0$ for all $x$ then $u_{xx} \ge 0$ for all $x$ and $t$. (Hint: differentiate the PDE.)

(b) Suppose $\bar{u}$ solves the analogous equation with $a$ replaced by $\bar{a} > a$, using the same final-time data $\Phi$. We continue to assume that $\Phi_{xx} \ge 0$. Show that $\bar{u} \ge u$ for all $x$ and $t$. (Hint: $w = \bar{u} - u$ solves $w_t + \bar{a} w_{xx} = f$ with $f = (a - \bar{a}) u_{xx} \le 0$.)