**PDE for Finance, Spring 2014 – Homework 4. Distributed 3/26/14, due 4/14/14.**
*Problem 3 corrected 4/8: parts (b) and (c) follow the lead of problem 2, not problem 1.*

Except for the first, these problems concern deterministic optimal control (Section 4 material). Some are a bit laborious (though not necessarily difficult). Moreover some students are still completing HW3. Therefore I have fixed the due date as 4/14 (almost 3 weeks).

(1) Consider the standard finite difference scheme

$$\frac{u_n^{m+1} - u_n^m}{\Delta t} = \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\Delta x)^2} \tag{1}$$

for solving $u_t - u_{xx} = 0$ on an interval. (Here $u_n^m$ is the numerical solution at time $m\Delta t$ and spatial location $n\Delta x$.) The stability restriction $\Delta t < \frac{1}{2}(\Delta x)^2$ leaves a lot of freedom in the choice of $\Delta x$ and $\Delta t$. Show that

$$\Delta t = \frac{1}{6}(\Delta x)^2$$

is special, in the sense that the numerical scheme (1) has errors of order $(\Delta x)^4$ rather than $(\Delta x)^2$. In other words: when $u$ is the exact solution of the PDE, the left and right sides of (1) differ by a term of order $(\Delta x)^4$. [Comment: the argument sketched in Section 3 shows that if $u$ solves the PDE and $v$ solves the finite difference scheme then $|u - v|$ is of order $(\Delta x)^2$ in general, but it is smaller – of order $(\Delta x)^4$ – when $\Delta t = \frac{1}{6}(\Delta x)^2$, provided that $u$ is sufficiently smooth.]

(2) Consider the finite-horizon utility maximization problem with discount rate $\rho$. The dynamical law is thus

$$dy/ds = f(y(s), \alpha(s)), \quad y(t) = x,$$

and the optimal utility discounted to time 0 is

$$u(x,t) = \max_{\alpha \in A} \left\{ \int_t^T e^{-\rho s} h(y(s), \alpha(s))\, ds + e^{-\rho T} g(y(T)) \right\}.$$

It is often more convenient to consider, instead of $u$, the optimal utility discounted to time $t$; this is

$$v(x,t) = e^{\rho t} u(x,t) = \max_{\alpha \in A} \left\{ \int_t^T e^{-\rho(s-t)} h(y(s), \alpha(s))\, ds + e^{-\rho(T-t)} g(y(T)) \right\}.$$

(a) Show (by a heuristic argument similar to those in the Section 4 notes) that $v$ satisfies

$$v_t - \rho v + H(x, \nabla v) = 0$$

with Hamiltonian

$$H(x,p) = \max_{a \in A} \{ f(x,a) \cdot p + h(x,a) \}$$

and final-time data

$$v(x,T) = g(x).$$

(Notice that the PDE for $v$ is autonomous, i.e. there is no explicit dependence on time.)

(b) Now consider the analogous infinite-horizon problem, with the same equation of state, and value function

$$\bar{v}(x,t) = \max_{\alpha \in A} \int_t^\infty e^{-\rho(s-t)} h(y(s), \alpha(s))\, ds.$$

Show (by an elementary comparison argument) that $\bar{v}$ is independent of $t$, i.e. $\bar{v} = \bar{v}(x)$ is a function of $x$ alone. Conclude using part (a) that if $\bar{v}$ is finite, it solves the stationary PDE

$$-\rho\bar{v} + H(x, \nabla\bar{v}) = 0.$$

(3) Recall Example 1 of the Section 4 notes: the state equation is $dy/ds = ry - \alpha$ with $y(t) = x$, and the value function is

$$u(x,t) = \max_{\alpha \geq 0} \int_t^\tau e^{-\rho s} h(\alpha(s))\, ds$$

with $h(a) = a^\gamma$ for some $0 < \gamma < 1$, and

$$\tau = \begin{cases} \text{first time when } y = 0 \text{ if this occurs before time } T \\ T \text{ otherwise.} \end{cases}$$

(a) We obtained a formula for $u(x,t)$ in the Section 4 notes, however our formula doesn't make sense when $\rho - r\gamma = 0$. Find the correct formula in that case.

(b) Let's examine the infinite-horizon-limit $T \to \infty$. Following the lead of Problem 2 let us concentrate on $v(x,t) = e^{\rho t} u(x,t)$ = optimal utility discounted to time $t$. Show that

$$\bar{v}(x) = \lim_{T\to\infty} v(x,t) = \begin{cases} G_\infty x^\gamma & \text{if } \rho - r\gamma > 0 \\ \infty & \text{if } \rho - r\gamma \leq 0 \end{cases}$$

with $G_\infty = [(1-\gamma)/(\rho - r\gamma)]^{1-\gamma}$.

(c) Use the stationary PDE of Problem 2(b) (specialized to this example) to obtain the same result.

(d) What is the optimal consumption strategy, for the infinite-horizon version of this problem?

(4) Consider the analogue of Example 1 with the power-law utility replaced by the logarithm: $h(a) = \ln a$. To avoid confusion let us write $u_\gamma$ for the value function obtained in the notes using $h(a) = a^\gamma$, and $u_{\log}$ for the value function obtained using $h(a) = \ln a$. Recall that $u_\gamma(x,t) = g_\gamma(t) x^\gamma$ with

$$g_\gamma(t) = e^{-\rho t} \left[ \frac{1-\gamma}{\rho - r\gamma} \left( 1 - e^{-\frac{(\rho - r\gamma)(T-t)}{1-\gamma}} \right) \right]^{1-\gamma}.$$

(a) Show, by a direct comparison argument, that

$$u_{\log}(\lambda x, t) = u_{\log}(x, t) + \frac{1}{\rho} e^{-\rho t}(1 - e^{-\rho(T-t)}) \ln \lambda$$

for any $\lambda > 0$. Use this to conclude that

$$u_{\log}(x, t) = g_0(t) \ln x + g_1(t)$$

where $g_0(t) = \frac{1}{\rho} e^{-\rho t}(1 - e^{-\rho(T-t)})$ and $g_1$ is an as-yet unspecified function of $t$ alone.

(b) Pursue the following scheme for finding $g_1$: Consider the utility $h = \frac{1}{\gamma}(a^\gamma - 1)$. Express its value function $u_h$ in terms of $u_\gamma$. Now take the limit $\gamma \to 0$. Show this gives a result of the expected form, with

$$g_0(t) = g_\gamma(t)|_{\gamma=0}$$

and

$$g_1(t) = \left. \frac{dg_\gamma}{d\gamma}(t) \right|_{\gamma=0}.$$

(This leads to an explicit formula for $g_1$ but it's messy; I'm not asking you to write it down.)

(c) Indicate how $g_0$ and $g_1$ could alternatively have been found by solving appropriate PDE's. (Hint: find the HJB equation associated with $h(a) = \ln a$, and show that the ansatz $u_{\log} = g_0(t) \ln x + g_1(t)$ leads to differential equations that determine $g_0$ and $g_1$.)

(5) Our Example 1 considers an investor who receives interest (at constant rate $r$) but no wages. Let's consider what happens if the investor also receives wages at constant rate $w$. The equation of state becomes

$$dy/ds = ry + w - \alpha \quad \text{with } y(t) = x,$$

and the value function is

$$u(x, t) = \max_{\alpha \geq 0} \int_t^T e^{-\rho s} h(\alpha(s))\, ds$$

with $h(a) = a^\gamma$ for some $0 < \gamma < 1$. Since the investor earns wages, we now permit $y(s) < 0$, however we insist that the final-time wealth be nonnegative ($y(T) \geq 0$).

(a) Which pairs $(x, t)$ are acceptable? The strategy that maximizes $y(T)$ is clearly to consume nothing ($\alpha(s) = 0$ for all $t < s < T$). Show this results in $y(T) \geq 0$ exactly if

$$x + \phi(t) w \geq 0$$

where

$$\phi(t) = \frac{1}{r}\left(1 - e^{-r(T-t)}\right).$$

Notice for future reference that $\phi$ solves $\phi' - r\phi + 1 = 0$ with $\phi(T) = 0$.

(b) Find the HJB equation that $u(x,t)$ should satisfy in its natural domain $\{(x,t) : x+\phi(t)w \geq 0\}$. Specify the boundary conditions when $t = T$ and where $x+\phi w = 0$.

(c) Substitute into this HJB equation the ansatz

$$v(x,t) = e^{-\rho t}G(t)(x+\phi(t)w)^\gamma.$$

Show $v$ is a solution when $G$ solves the familiar equation

$$G_t + (r\gamma - \rho)G + (1-\gamma)G^{\gamma/(\gamma-1)} = 0$$

(the same equation we solved in Example 1). Deduce a formula for $v$.

(d) In view of (a), a more careful definition of the value function for this control problem is

$$u(x,t) = \max_{\alpha \geq 0} \int_t^\tau e^{-\rho s} h(\alpha(s))\, ds$$

where

$$\tau = \begin{cases} \text{first time when } y(s) + \phi(s)w = 0 \text{ if this occurs before time } T \\ T \text{ otherwise.} \end{cases}$$

Use a verification argument to prove that the function $v$ obtained in (c) is indeed the value function $u$ defined this way.

(6) This problem is a special case of the "linear-quadratic regulator" widely used in engineering applications. The state is $y(s) \in R^n$, and the control is $\alpha(s) \in R^n$. There is no pointwise restriction on the values of $\alpha(s)$. The evolution law is

$$dy/ds = Ay + \alpha, \quad y(t) = x,$$

for some constant matrix $A$, and the goal is to minimize

$$\int_t^T |y(s)|^2 + |\alpha(s)|^2\, ds + |y(T)|^2.$$

(In words: we prefer $y = 0$ along the trajectory and at the final time, but we also prefer not to use too much control.)

(a) What is the associated Hamilton-Jacobi-Bellman equation? Explain why we should expect the relation $\alpha(s) = -\frac{1}{2}\nabla u(y(s))$ to hold along optimal trajectories.

(b) Since the problem is quadratic, it's natural to guess that the value function $u(x,t)$ takes the form

$$u(x,t) = \langle K(t)x, x\rangle$$

for some symmetric $n \times n$ matrix-valued function $K(t)$. Show that this $u$ solves the Hamilton-Jacobi-Bellman equation exactly if

$$\frac{dK}{dt} = K^2 - I - (K^T A + A^T K) \text{ for } t < T, \quad K(T) = I$$

where $I$ is the $n \times n$ identity matrix. (Hint: two quadratic forms agree exactly if the associated symmetric matrices agree.)

(c) Show by a suitable verification argument that this $u$ is indeed the value function of the control problem.