PDE for Finance, Spring 2014 – Homework 6
Distributed 4/29/14, due 5/12/14.

(1) The Section 7 notes discuss work by Bertsimas, Kogan, and Lo involving least-square replication of a European option. The analysis there assumes all trades are *self-financing*, so the value of the portfolio at consecutive times is related by

$$V_j - V_{j-1} = \theta_{j-1}(P_j - P_{j-1}).$$

Let's consider what happens if trades are permitted to be non-self-financing. This means we introduce an additional control g_j, the amount of cash added to (if $g_j > 0$) or removed from (if $g_j < 0$) the portfolio at time j, and the portfolio values now satisfy

$$V_j - V_{j-1} = \theta_{j-1}(P_j - P_{j-1}) + g_{j-1}.$$

It is natural to add a quadratic expression involving the g's to the objective: now we seek to minimize

$$E\left[(V_N - F(P_N))^2 + \alpha \sum_{j=0}^{N-1} g_j^2\right]$$

where α is a positive constant. The associated value function is

$$J_i(V,P) = \min_{\theta_i, g_i; \ldots; \theta_{N-1}, g_{N-1}} E_{V_i=V,\, P_i=P}\left[(V_N - F(P_N))^2 + \alpha \sum_{j=i}^{N-1} g_j^2\right].$$

The claim enunciated in the Section 7 notes remains true in this modified setting: J_i can be expressed as a quadratic polynomial

$$J_i(V_i, P_i) = \bar{a}_i(P_i)|V_i - \bar{b}_i(P_i)|^2 + \bar{c}_i(P_i)$$

where $\bar{a}_i, \bar{b}_i$, and $\bar{c}_i$ are suitably-defined functions which can be constructed inductively. Demonstrate this assertion in the special case $i = N-1$, and explain how $\bar{a}_{N-1}, \bar{b}_{N-1}, \bar{c}_{N-1}$ are related to the functions $a_{N-1}, b_{N-1}, c_{N-1}$ of the Section 7 notes.

(2) Consider scaled Brownian motion with drift and jumps: $dy = \mu dt + \sigma dw + JdN$, starting at $y(0) = 0$. Assume the jump occurences are Poisson with rate λ, and the jump magnitudes J are Gaussian with mean 0 and variance δ^2. Find the probability distribution of the process y at time t. (*Hint*: don't try to use the forward equation. Instead observe that you know, for any n, the probability that n jumps will occur before time t; and after conditioning on the number of jumps, the distribution of y is a Gaussian whose mean and variance are easy to determine. Assemble these ingredients to give the density of y as an infinite sum.)