PDE for Finance, 4/28/2014

Discuss today why "risk neutral expected payoff" can be used to price (even path-dependent) options.

To keep things concrete I'll focus on our usual classes of (simple) market models: lognormal with constant interest rate r, or else $dS = \mu(S,t)S\,dt + \sigma(S,t)S\,dw$ (still Markovian), however the discn extends far beyond that (even to $dS = \mu_t S\,dt + \sigma_t S\,dw$ where μ_t, σ_t are just $\mathcal{F}_t$ measurable, and even to stoch interest rates)

Places to read about this:

- my 2004 Cont's Time Finance notes, Sections 1 + 2,
- the book by Baxter + Rennie
- the book by Korn + Korn

Big picture: we have thus far understood only the "Feynman-Kac" leg of the following triangle

BS pde $\xrightarrow{\text{Ito}}$ replicating portfolio

Feynman-Kac — Martingale repn. + Girsanov

discounted risk-neutral expectation

First, relatively easy goal: explain the "Ito" leg. This applies only to options whose payoff has the form $\phi(S(T))$ (no path dependence), since the Black-Scholes PDE is restricted to that setting.

Claim: if $V(S,t)$ solves BS pde with $V=\phi$ at $t=T$, then starting with initial capital $V(S_0,0)$ at time 0, option's payoff can be replicated by a self-financing trading strategy. [So (by absence of arbitrage) option's value at time 0 must be $V(S_0,0)$.]

Pf of claim: consider trading strategy:

- ~~at~~ at time t hold $\phi_t = \frac{\partial V}{\partial S}(S_t,t)$ units of stock and $\psi_t = (V(S_t,t) - \phi_t S_t)/B_t$ units of bond

where (since we take the interest rate to be constant) $B_t = e^{rt}$ is the value of a risk-free bond at time t.

Clearly its total value is $V(S_t, t)$. To show it is self-financing we must show that

$$\underbrace{dV}_{\text{change in value}} = \underbrace{\varphi_t\, dS + \psi_t\, dB}_{\text{profit or loss on stock + bond holdings}}$$

LHS is $dV = V_t\,dt + V_S\,dS + \frac{1}{2}V_{SS}\,dS\,dS$ by Ito,

RHS is $\frac{\partial V}{\partial S}dS + \left(V - \frac{\partial V}{\partial S}S\right)r$ since $dB = rB\,dt$

The fact that these are equal is precisely the BS pde.

Of course we know that BS pde is assoc to Feynman-Kac applied to $dS = rS\,dt + \sigma S\,dW$. So we recover this way that option prices are assoc to discounted expected payoffs wrto a suitable stock process (the "risk-neutral process"), different from the subjective one.

But: this applies only to path-independent options. I'd like a viewpt that applies also to path-dependent options (ie any $\mathcal{F}_1$-measurable payoff).

Among other things, we need this for discn (next wk) of the "martingale approach" to portfolio optimization.

A key fact (Girsanov's thm): changing the drift of an SDE amounts to changing the measure we consider on path space (and the Radon-Nikodym derivative can be made explicit). In fact, if

$$dS = \mu S\,dt + \sigma S\,dW$$ under the original measure, call it P.

Then there's a different measure, call it Q, st

$$dS = rS\,dt + \sigma S\,d\tilde{W}$$ where $\tilde{W}$ is a Q-Brownian motion

(so: $d\tilde{W} = \frac{\mu-r}{\sigma}dt + dW$) and for every $\mathcal{F}_T$-meas random var X we have

$$E_Q[X] = E_P[M_T X]$$

with

$$M_t = \exp\left(-\int_0^t \left(\frac{\mu-r}{\sigma}\right)dW - \frac{1}{2}\int_0^t \left(\frac{\mu-r}{\sigma}\right)^2 ds\right).$$

(This is Girsanov's thm specialized to our simple setting; of course the theorem applies much more generally.)

In short: under the measure Q, S_t/B_t is a martingale (since $d(S/B) = d(e^{-rt}S) = dS - rS\,dt$
$= \sigma S\, d\tilde{W}$)

Why is this useful? I claim that the value V_t (at time t) of a pathdependent option worth X at time T satisfies

$$(*) \quad V_t/B_t = E_Q\left[X/B_T \mid \mathcal{F}_t \right].$$

where the RHS is the conditional expectation wrt to info available at time t.

To prove this, we must show (as we did earlier) existence of a replicating portfolio that's self financing, whose value at time t is the V_t defined above.

Key ingredient of pf: we already know that if $dy = \gamma\, d\tilde{W}$ then y is a Q-martingale. The converse is also true: every martingale has this form for some γ, so (using $d(S/B) = \sigma S\, d\tilde{W}$)

$$(*) \quad \Rightarrow \quad d(V_t/B_t) = \varphi_t\, d(S/B_t)$$

for some ($\mathcal{F}_t$ - meas) φ_t.

We use this φ_t to construct the trading strategy:

- repl portfolio holds φ_t units of stock and $(V_t - \varphi_t S_t)/B_t$ units of bond at time t.

Its value at time t is V_t. Is it self-financing? Well,

$$V = \frac{V}{B} B \Rightarrow dV = d\left(\frac{V}{B}\right) \cdot B + \frac{V}{B} dB.$$

(I'm using Ito in form $d(XY) = X\,dY + Y\,dX + dX\,dY$, and fact that $dX\,dB = 0$ since $dB = rB\,dt$ has no "dW" term.)

And

$$S = \frac{S}{B} B \Rightarrow dS = d(S/B) B + \frac{S}{B} dB$$

so

$$\varphi\, dS + \psi\, dB = \varphi B\, d(S/B) + \cancel{\varphi \frac{S}{B} dB} + \left(\frac{V}{B} - \cancel{\varphi \frac{S}{B}}\right) dB$$

So the choice of φ s.t. $\varphi\, d(S/B) = d(V/B)$ assures the portfolio is self-financing.

Rmk: since we know how to turn Q-expectations into P-expectations (using Girsanov) we can also write down the value of an option using the P-expectation.

Some examples, to bring this down to earth

① stock with const dividend yield at rate q. The tradeable in this case is the stock with dividends reinvested.

Claim: if stock price process (under P) is

$$dS = \mu S\,dt + \sigma S\,dW$$

then RN process (assoc Q) is

$$(**) \qquad dS = (r-q) S\,dt + \sigma S\,d\tilde{W} .$$

The reason is that if you start at $t=0$ with 1 share then your holding at time t is e^{qt} shares, and its value is $S_t e^{qt}$. Eqn (**) assures that

$$S_t e^{qt}/B_t = S_t e^{(q-r)t} \text{ is a Q-martingale.}$$

② Option on a foreign exchange rate

Suppose US dollar risk-free rate $= r$
" British pound risk-free rate $= q$
exchange rate (dollars/pound) is lognormal,
$dC = \mu C\,dt + \sigma C\,dw$.

To dollar investor, a pound looks like "stock with cont's div yield q", so from ex ① the dollar-investor's risk-neutral process is Q, where

$$dC = (r-q)\,C\,dt + \sigma C\,d\tilde{w}$$ & $d\tilde{w}$ is a Q-Brownian motion.

What about the pound investor. His exchange rate is $1/C$. By Ito, under the P measure,

$$d(1/C) = -C^{-2}dC + C^{-3}dC\,dC$$
$$= (-\mu+\sigma^2)\frac{1}{C}dt - \sigma\cdot\frac{1}{C}dw$$

What is the Pound investor's risk-neutral measure? Certainly not Q! By analogy to what we did for the dollar investor, pound investor's RN meas $\bar{Q}$ is st

$$d(1/C) = (q-r)(1/C)\,dt - \sigma(1/C)\,d\bar{w}$$

where $\bar{W}$ is a $\bar{Q}$ Brownian motion. Evidently,

$$(-\mu+\sigma^2)\,dt - \sigma\,dW = (q-r)\,dt - \sigma\,d\bar{W}$$

$$\text{whence} \quad d\bar{W} = \left(\frac{\mu+q-r}{\sigma} - \sigma\right)dt + dW$$

whereas for the dollar investor

$$\mu \cancel{C}\,dt + \sigma \cancel{C}\,dW = (r-q)\cancel{C}\,dt + \sigma \cancel{C}\,d\tilde{W}$$

$$\Rightarrow d\tilde{W} = \left(\frac{\mu+q-r}{\sigma}\right)dt + dW$$

Is this strange? Well, no. The "RN measure" is simply the one assuring that (value of tradable/B_t) is a martingale. Different investors in this case see different tradables, so they have different RN measures.