PDE for Finance, Spring 2015 – Homework 4. Distributed 3/31/15, due 4/13/15.

(1) Consider the standard finite difference scheme

$$\frac{u_n^{m+1} - u_n^m}{\Delta t} = \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\Delta x)^2} \tag{1}$$

for solving $u_t - u_{xx} = 0$ on an interval. (Here u_n^m is the numerical solution at time $m\Delta t$ and spatial location $n\Delta x$.) The stability restriction $\Delta t < \frac{1}{2}(\Delta x)^2$ leaves a lot of freedom in the choice of Δx and Δt. Show that

$$\Delta t = \frac{1}{6}(\Delta x)^2$$

is special, in the sense that the numerical scheme (1) has errors of order $(\Delta x)^4$ rather than $(\Delta x)^2$. In other words: when u is the exact solution of the PDE, the left and right sides of (1) differ by a term of order $(\Delta x)^4$. [Comment: the argument sketched in Section 3 shows that if u solves the PDE and v solves the finite difference scheme then $|u - v|$ is of order $(\Delta x)^2$ in general, but it is smaller – of order $(\Delta x)^4$ – when $\Delta t = \frac{1}{6}(\Delta x)^2$, provided that u is sufficiently smooth.]

(2) Consider the finite-horizon utility maximization problem with discount rate ρ. The dynamical law is thus

$$dy/ds = f(y(s), \alpha(s)), \quad y(t) = x,$$

and the optimal utility discounted to time 0 is

$$u(x,t) = \max_{\alpha \in A} \left\{ \int_t^T e^{-\rho s} h(y(s), \alpha(s))\, ds + e^{-\rho T} g(y(T)) \right\}.$$

It is often more convenient to consider, instead of u, the optimal utility discounted to time t; this is

$$v(x,t) = e^{\rho t} u(x,t) = \max_{\alpha \in A} \left\{ \int_t^T e^{-\rho(s-t)} h(y(s), \alpha(s))\, ds + e^{-\rho(T-t)} g(y(T)) \right\}.$$

(a) Show (by a heuristic argument similar to those in the Section 4 notes) that v satisfies

$$v_t - \rho v + H(x, \nabla v) = 0$$

with Hamiltonian

$$H(x,p) = \max_{a \in A} \{ f(x,a) \cdot p + h(x,a) \}$$

and final-time data

$$v(x,T) = g(x).$$

(Notice that the PDE for v is autonomous, i.e. there is no explicit dependence on time.)

(b) Now consider the analogous infinite-horizon problem, with the same equation of state, and value function

$$\bar{v}(x,t) = \max_{\alpha \in A} \int_t^\infty e^{-\rho(s-t)} h(y(s), \alpha(s))\, ds.$$

Show (by an elementary comparison argument) that $\bar{v}$ is independent of t, i.e. $\bar{v} = \bar{v}(x)$ is a function of x alone. Conclude using part (a) that if $\bar{v}$ is finite, it solves the stationary PDE

$$-\rho\bar{v} + H(x, \nabla\bar{v}) = 0.$$

(3) Recall Example 1 of the Section 4 notes: the state equation is $dy/ds = ry - \alpha$ with $y(t) = x$, and the value function is

$$u(x,t) = \max_{\alpha \geq 0} \int_t^\tau e^{-\rho s} h(\alpha(s))\, ds$$

with $h(a) = a^\gamma$ for some $0 < \gamma < 1$, and

$$\tau = \begin{cases} \text{first time when } y = 0 \text{ if this occurs before time } T \\ T \text{ otherwise.} \end{cases}$$

(a) We obtained a formula for $u(x,t)$ in the Section 4 notes, however our formula doesn't make sense when $\rho - r\gamma = 0$. Find the correct formula in that case.

(b) Let's examine the infinite-horizon-limit $T \to \infty$. Following the lead of Problem 2 let us concentrate on $v(x,t) = e^{\rho t} u(x,t)$ = optimal utility discounted to time t. Show that

$$\bar{v}(x) = \lim_{T\to\infty} v(x,t) = \begin{cases} G_\infty x^\gamma & \text{if } \rho - r\gamma > 0 \\ \infty & \text{if } \rho - r\gamma \leq 0 \end{cases}$$

with $G_\infty = [(1-\gamma)/(\rho - r\gamma)]^{1-\gamma}$.

(c) Use the stationary PDE of Problem 2(b) (specialized to this example) to obtain the same result.

(d) What is the optimal consumption strategy, for the infinite-horizon version of this problem?

(4) Consider the analogue of Example 1 with the power-law utility replaced by the logarithm: $h(a) = \ln a$. To avoid confusion let us write u_γ for the value function obtained in the notes using $h(a) = a^\gamma$, and $u_{\log}$ for the value function obtained using $h(a) = \ln a$. Recall that $u_\gamma(x,t) = g_\gamma(t) x^\gamma$ with

$$g_\gamma(t) = e^{-\rho t} \left[\frac{1-\gamma}{\rho - r\gamma} \left(1 - e^{-\frac{(\rho - r\gamma)(T-t)}{1-\gamma}} \right) \right]^{1-\gamma}.$$

(a) Show, by a direct comparison argument, that

$$u_{\log}(\lambda x, t) = u_{\log}(x, t) + \frac{1}{\rho} e^{-\rho t}(1 - e^{-\rho(T-t)}) \ln \lambda$$

for any $\lambda > 0$. Use this to conclude that

$$u_{\log}(x, t) = g_0(t) \ln x + g_1(t)$$

where $g_0(t) = \frac{1}{\rho} e^{-\rho t}(1 - e^{-\rho(T-t)})$ and g_1 is an as-yet unspecified function of t alone.

(b) Pursue the following scheme for finding g_1: Consider the utility $h = \frac{1}{\gamma}(a^\gamma - 1)$. Express its value function u_h in terms of u_γ. Now take the limit $\gamma \to 0$. Show this gives a result of the expected form, with

$$g_0(t) = g_\gamma(t)|_{\gamma=0}$$

and

$$g_1(t) = \frac{dg_\gamma}{d\gamma}(t)\bigg|_{\gamma=0}.$$

(This leads to an explicit formula for g_1 but it's messy; I'm not asking you to write it down.)

(c) Indicate how g_0 and g_1 could alternatively have been found by solving appropriate PDE's. (Hint: find the HJB equation associated with $h(a) = \ln a$, and show that the ansatz $u_{\log} = g_0(t) \ln x + g_1(t)$ leads to differential equations that determine g_0 and g_1.)

(5) Our Example 1 considers an investor who receives interest (at constant rate r) but no wages. Let's consider what happens if the investor also receives wages at constant rate w. The equation of state becomes

$$dy/ds = ry + w - \alpha \quad \text{with } y(t) = x,$$

and the value function is

$$u(x, t) = \max_{\alpha \geq 0} \int_t^T e^{-\rho s} h(\alpha(s))\, ds$$

with $h(a) = a^\gamma$ for some $0 < \gamma < 1$. Since the investor earns wages, we now permit $y(s) < 0$, however we insist that the final-time wealth be nonnegative ($y(T) \geq 0$).

(a) Which pairs (x, t) are acceptable? The strategy that maximizes $y(T)$ is clearly to consume nothing ($\alpha(s) = 0$ for all $t < s < T$). Show this results in $y(T) \geq 0$ exactly if

$$x + \phi(t) w \geq 0$$

where

$$\phi(t) = \frac{1}{r}\left(1 - e^{-r(T-t)}\right).$$

Notice for future reference that ϕ solves $\phi' - r\phi + 1 = 0$ with $\phi(T) = 0$.

(b) Find the HJB equation that $u(x,t)$ should satisfy in its natural domain $\{(x,t) : x+\phi(t)w \geq 0\}$. Specify the boundary conditions when $t = T$ and where $x+\phi w = 0$.

(c) Substitute into this HJB equation the ansatz

$$v(x,t) = e^{-\rho t} G(t)(x + \phi(t)w)^{\gamma}.$$

Show v is a solution when G solves the familiar equation

$$G_t + (r\gamma - \rho)G + (1-\gamma)G^{\gamma/(\gamma-1)} = 0$$

(the same equation we solved in Example 1). Deduce a formula for v.

(d) In view of (a), a more careful definition of the value function for this control problem is

$$u(x,t) = \max_{\alpha \geq 0} \int_t^{\tau} e^{-\rho s} h(\alpha(s))\, ds$$

where

$$\tau = \begin{cases} \text{first time when } y(s) + \phi(s)w = 0 \text{ if this occurs before time } T \\ T \text{ otherwise.} \end{cases}$$

Use a verification argument to prove that the function v obtained in (c) is indeed the value function u defined this way.

(6) This problem is a special case of the "linear-quadratic regulator" widely used in engineering applications. The state is $y(s) \in R^n$, and the control is $\alpha(s) \in R^n$. There is no pointwise restriction on the values of $\alpha(s)$. The evolution law is

$$dy/ds = Ay + \alpha, \quad y(t) = x,$$

for some constant matrix A, and the goal is to minimize

$$\int_t^T |y(s)|^2 + |\alpha(s)|^2\, ds + |y(T)|^2.$$

(In words: we prefer $y = 0$ along the trajectory and at the final time, but we also prefer not to use too much control.)

(a) What is the associated Hamilton-Jacobi-Bellman equation? Explain why we should expect the relation $\alpha(s) = -\frac{1}{2}\nabla u(y(s))$ to hold along optimal trajectories.

(b) Since the problem is quadratic, it's natural to guess that the value function $u(x,t)$ takes the form

$$u(x,t) = \langle K(t)x, x\rangle$$

for some symmetric $n \times n$ matrix-valued function $K(t)$. Show that this u solves the Hamilton-Jacobi-Bellman equation exactly if

$$\frac{dK}{dt} = K^2 - I - (K^T A + A^T K) \text{ for } t < T, \quad K(T) = I$$

where I is the $n \times n$ identity matrix. (Hint: two quadratic forms agree exactly if the associated symmetric matrices agree.)

(c) Show by a suitable verification argument that this u is indeed the value function of the control problem.