

Concavity and Expected Return, 11/3/03

I've recommended P. Kritzman, *Puzzles of Finance: Six Practical Problems and their Remarkable Solutions* (John Wiley & Sons, paper, about \$20). Chapter 4 is entitled "Why the expected return is not to be expected." These notes provide a brief discussion what he means. (They are not however the same as his chapter – which makes delightful reading.)

Let R be the return on some specific type of investment. (To fix ideas: it could be the annual return on a stock index such as the S&P 500 or the Russell 3000.) We take the view, as usual, that R is a random variable, and each year represents an independent sample.

The *expected return* is, by definition, $E[R]$. In practice we would estimate this by taking the "sample mean," i.e. considering the arithmetic average of the returns $R_1, \dots, R_N$ experienced over N consecutive recent years:

$$\text{expected return} = E[R] \approx \frac{R_1 + \dots + R_N}{N}$$

where the approximation is good when N is large.

The return achievable by a multiperiod buy-and-hold investor is different. It is what we've called the *long-term-equivalent return* $R_{\text{LTE}} = \exp(E[\log R])$. In practice we would estimate this by taking the sample mean of $\log R$:

$$R_{\text{LTE}} = e^{E[\log R]} \approx e^{\frac{\log R_1 + \dots + \log R_N}{N}} = (R_1 \dots R_N)^{1/N} = R_{\text{ann}}$$

where the approximation is good when N is large. The notation R_{ann} refers to the *annualized return*, defined for a given N -year period by $R_{\text{ann}} = (R_1 R_2 \dots R_N)^{1/N}$. This is the fixed return which, compounded over N years, gives the same result as the experienced returns $R_1 R_2 \dots R_N$.

The expected return is not to be expected, because the long-term-equivalent return is always *less than* the expected return. In other words,

$$e^{E[\log R]} < E[R].$$

This is a consequence of Jensen's inequality, which says $E[f(X)] \leq f(E[X])$ when X is a random variable and f is concave. (The recent handout on concavity gives a proof.) Actually, we need a slightly sharper result: the inequality is strict, i.e. $E[f(X)] < f(E[X])$, provided f is strictly concave ($f'' < 0$) and X is truly random ($\text{Var}(X) > 0$). (This is easily seen by examining the proof of Jensen's inequality.) Applying this to $X = R$ and $f = \log$ gives

$$E[\log R] < \log E[R],$$

from which it follows that

$$e^{E[\log R]} < e^{\log E[R]} = E[R]$$

as asserted.

What's going on? Briefly, the point is that a buy-and-hold investor experiences *compound* interest, which involves geometric rather than arithmetic averages of the annual returns.

Let's explain this. In the process, we'll see that the probabilistic language used above isn't really necessary. Also, we'll see that the phenomenon isn't restricted to one's experience over long periods of time; it can be seen in the results over any period of N years, even when N is small (say, $N = 2$ or 3).

Here's the point: if $R_1, \dots, R_N$ are N consecutive years' returns, then

$$\begin{aligned}\text{annualized return} &= (R_1 \cdots R_N)^{1/N} \\ \text{arithmetic average return} &= (R_1 + \cdots + R_N)/N.\end{aligned}$$

The annualized return is also sometimes called the "geometric average return." It is always *less* than the arithmetic average return, as a consequence of the "arithmetic mean – geometric mean inequality," which says

$$\frac{x_1 + \cdots + x_N}{N} \geq (x_1 \cdots x_N)^{1/N}$$

for any numbers positive numbers $x_1, \dots, x_N$, with strict inequality unless all the x_i 's are equal. (This is easy to prove by direct calculation when $N = 2$, but such a proof for $N = 3$ is already awkward. However the arithmetic mean – geometric mean inequality is an easy consequence of Jensen's inequality: consider the random variable X taking each value x_i with probability $1/N$, and apply Jensen to $f(X) = \log X$.)

Now let's return to the probabilistic viewpoint, and consider the *probability distribution* of annualized returns. To keep things simple, we'll assume that the statistics of the single-year returns are very simple:

$$R = \begin{cases} 1.25 & \text{with probability } 1/2 \\ .75 & \text{with probability } 1/2. \end{cases}$$

Notice that $E[R] = 1$.

Over a period of two years, a buy-and-hold investor experiences

$$R_1 R_2 = \begin{cases} (5/4)^2 = 25/16 & \text{with probability } 1/4 \\ (5/4)(3/4) = 15/16 & \text{with probability } 1/2 \\ (3/4)^2 = 9/16 & \text{with probability } 1/4. \end{cases}$$

The phenomenon discussed above is naturally present in this example. Since distinct years are independent we have $E[R_1 R_2] = E[R_1]E[R_2] = 1$. Jensen's inequality tells us that $E[R^{1/2}] < 1$ (since $f(R) = R^{1/2}$ is concave). Using independence again we see that *expected annualized return is less than the expected return*:

$$E[(R_1 R_2)^{1/2}] = E[(R_1)^{1/2}]E[(R_2)^{1/2}] < 1.$$

(This can also be verified by direct calculation, using the probability distribution of $R_1 R_2$.)

But looking at the statistics of $R_1 R_2$ we see something more: *most investors do worse than the expected return*. That is: with probability $3/4$, the annualized return R_{ann} is less than

$E[R] = 1$. This captures a general fact, present (indeed, more pronounced) also when N is large. By independence we know

$$E[R_1 \cdots R_N] = 1$$

since $E[R] = 1$. However most investors experience an overall return $R_1 \cdots R_N$ that's substantially below 1. Isn't this a contradiction? No – a few lucky investors experience an overall return $R_1 \cdots R_N$ that's quite large, and they bring the mean behavior up to 1. In summary: the probability distribution of $R_1 \cdots R_N$ has mean 1, but it is not symmetric about the mean. Rather, most of its “mass” is below 1.

To understand this concretely, consider the experience of a buy-and-hold investor over 50 years. What's the probability that $R_{\text{ann}} > E[R]$? In other words, what is the probability of achieving $R_1 \cdots R_{50} \geq 1$? This is easy to calculate in our example. If the stock goes up k times and down $50 - k$ times then

$$R_1 \cdots R_{50} = \left(\frac{5}{4}\right)^k \left(\frac{3}{4}\right)^{50-k}.$$

One verifies that this product is less than 1 when $k \leq 28$, and larger than 1 when $k \geq 29$. So

Prob that $R_1 \cdots R_{50} \geq 1$ = Prob of 29 or more heads in 50 coin-flips.

Now, the probability of getting heads exactly k times in 50 flips is $2^{-k} \binom{50}{k}$. Evaluating this by the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is not very convenient because both the numerator and denominator are typically very large. Using Pascal's triangle is better. But Excel or Matlab can do it for you (in Excel use the function BINOMDIST, in Matlab use the function NCHOOSEK). One finds that

$$\begin{aligned} \text{prob of 25 heads in 50 flips} &= .1123 \\ \text{prob of 26 heads in 50 flips} &= .1080 \\ \text{prob of 27 heads in 50 flips} &= .0960 \\ \text{prob of 28 heads in 50 flips} &= .0788. \end{aligned}$$

By symmetry, the probability of getting 26 or more heads is $(1 - .1123)/2 = .4438$. So the probability of getting 29 or more heads is $.4438 - .1080 - .0960 - .0788 = .1610$.

Our simple example (with R taking just two possible values in each period) has the advantage that we understand the distribution of $R_1 \cdots R_N$ exactly, since it is determined by N independent coin flips. So we can answer almost any question about the annual return. For example, what is the median annual return over 50 years? Answer: since the median number of heads from 50 flips is 25, the median annual return is

$$\left[\left(\frac{5}{4}\right)^{25} \left(\frac{3}{4}\right)^{25} \right]^{1/50} = \left(\frac{5}{4}\right)^{1/2} \left(\frac{3}{4}\right)^{1/2} = .9682.$$