

# ENERGY CRITICAL NLS IN TWO SPACE DIMENSIONS

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ABSTRACT. We investigate the initial value problem for a defocusing nonlinear Schrödinger equation with exponential nonlinearity

$$i\partial_t u + \Delta u = u(e^{4\pi|u|^2} - 1) \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x^2.$$

We identify subcritical, critical and supercritical regimes in the energy space. We establish global well-posedness in the subcritical and critical regimes. Well-posedness fails to hold in the supercritical case.

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1. WHAT IS THE ENERGY CRITICAL NLS EQUATION ON  $\mathbb{R}^2$ ?

We consider the initial value problem for a defocusing nonlinear Schrödinger equation with exponential nonlinearity

$$(1.1) \quad \begin{cases} i\partial_t u + \Delta u = f(u), & u : (-T_*, T^*) \times \mathbb{R}^2 \mapsto \mathbb{C} \\ u(0, \cdot) = u_0(\cdot) \in H^1(\mathbb{R}^2) \end{cases}$$

where

$$(1.2) \quad f(u) = u(e^{4\pi|u|^2} - 1).$$

Solutions of (1.1) formally satisfy the conservation of mass and Hamiltonian

$$(1.3) \quad \begin{aligned} M(u(t, \cdot)) &:= \|u(t, \cdot)\|_{L^2}^2 \\ &= M(u(0, \cdot)), \end{aligned}$$

$$(1.4) \quad \begin{aligned} H(u(t, \cdot)) &:= \left\| \nabla u(t, \cdot) \right\|_{L^2}^2 + \frac{1}{4\pi} \left\| e^{4\pi|u(t, \cdot)|^2} - 1 - 4\pi|u(t, \cdot)|^2 \right\|_{L^1(\mathbb{R}^2)} \\ &= H(u(0, \cdot)). \end{aligned}$$

We show that for initial data  $u_0$  satisfying  $H(u_0) \leq 1$  the initial value problem is global-in-time well-posed. Well-posedness fails to hold for data satisfying  $H(u_0) > 1$ . We compare our theory for (1.1) with work on the  $\dot{H}^1$  critical NLS initial value problem on  $\mathbb{R}^d$  with  $d \geq 3$ . Similar ill-posedness results were also obtained for the nonlinear Klein-Gordon equation with exponential nonlinearity in [21].

**1.1.  $NLS_p(\mathbb{R}^d)$  and critical regularity for local well-posedness.** We introduce a family of equations and identify (1.1) as a natural extreme limit of the family with monomial (or polynomial) nonlinearities when the space dimension is 2. The monomial defocusing semilinear initial value problem

$$(1.5) \quad \begin{cases} i\partial_t u + \Delta u = |u|^{p-1}u, & u : (-T_*, T^*) \times \mathbb{R}^d \mapsto \mathbb{C} \\ u(0, x) = u_0(x) \end{cases}$$

has solutions which also satisfy conservation of mass and Hamiltonian, where

$$(1.6) \quad H_p(u(t, \cdot)) := \|\nabla u(t, \cdot)\|_{L^2}^2 + \int_{\mathbb{R}^d} \frac{1}{p+1} |u|^{p+1}(t, x) dx.$$

We will sometimes refer to the initial value problem (1.5) with the notation  $NLS_p(\mathbb{R}^d)$ .

If  $u$  solves (1.5) then, for  $\lambda > 0$ ,  $u^\lambda : (-T_*\lambda^2, T^*\lambda^2) \times \mathbb{R}^d$  defined by

$$(1.7) \quad u^\lambda(t, x) := \lambda^{2/(1-p)} u(\lambda^{-2}t, \lambda^{-1}x)$$

also solves (1.5). It turns out that Banach spaces whose norms are invariant under the dilation  $u \mapsto u^\lambda$  are relevant in the theory of the initial value problem (1.5). Let  $s_c = \frac{d}{2} - \frac{2}{p-1}$ . Note that for all  $\lambda > 0$  the  $L^2$ -based homogeneous  $\dot{H}^{s_c}$  Sobolev norm is invariant under the the mapping  $f(x) \mapsto \lambda^{-2/(p-1)} f(\lambda^{-1}x)$ . Similarly, note that the Lebesgue  $L^{p_c}(\mathbb{R}_x^d)$  norm is invariant under the dilation symmetry for

$p_c = \frac{d}{2}(p-1)$ . Unless otherwise stated, we will restrict<sup>1</sup> this discussion to problems where dimension  $d$  and the degree  $p$  are constrained to give  $0 \leq s_c \leq 1$ . It is now known ([9], [18], [11]) that (1.5) with  $H^s$  initial data is locally well-posed for  $s > s_c$  with existence interval depending only upon  $\|u_0\|_{H^s}$ , locally well-posed for  $s = s_c$  with existence interval depending upon  $e^{it\Delta}u_0$ , and is ill-posed for  $s < s_c$ . Based on this complete trichotomy, it is natural to refer to  $H^{s_c}$  as the *critical regularity* for (1.5).

**1.2. Global well-posedness for  $NLS_p(\mathbb{R}^d)$ .** For the *energy subcritical* case, when  $s_c < 1$ , an iteration of the local-in-time well-posedness result using the *a priori* upper bound on  $\|u(t)\|_{H^1}$  implied by the conservation laws establishes global well-posedness for (1.5) in  $H^1$ . It is expected that the local-in-time  $H^{s_c}$  solutions of (1.5) extend to global-in-time solutions. For certain choices of  $p, d$  in the energy subcritical case, there are results ([2], [3], [15], [38], [16]) which establish that  $H^s$  initial data  $u_0$  evolve into global-in-time solutions  $u$  of (1.5) for  $s \in (\tilde{s}_{p,d}, 1)$  with  $s_c < \tilde{s}_{p,d} < 1$  and  $\tilde{s}_{p,d}$  close to 1 and away from  $s_c$ . For all problems with  $0 \leq s_c < 1$ , global well-posedness in the scaling invariant space  $H^{s_c}$  is unknown but conjectured to hold.

For the *energy critical* case, when  $s_c = 1$ , an iteration of the local-in-time well-posedness theory fails to prove global well-posedness. Since the local-in-time existence interval depends upon absolute continuity properties of the linear evolution  $e^{it\Delta}u_0$  (and not upon the controlled norm  $\|u(t)\|_{H^1}$ ), the local theory does not directly globalize based on the conservation laws. Nevertheless, based on new ideas of Bourgain in [3] (see also [4]) (which treated the radial case in dimension 3) and a new interaction Morawetz inequality [16] the energy critical case of (1.5) is now completely resolved [14, 39, 33] : Finite energy initial data  $u_0$  evolve into global-in-time solutions  $u$  with finite spacetime size  $\|u\|_{L_{t,x}^{\frac{2(2+d)}{d-2}}} < \infty$  and scatter.

**1.3. Energy criticality in two space dimensions.** The initial value problem  $NLS_p(\mathbb{R}^2)$  is energy subcritical for all  $p > 1$ . To identify an "energy critical" nonlinear Schrödinger initial value problem on  $\mathbb{R}^2$ , it is thus natural to consider problems with exponential nonlinearities. In this paper, we establish local and global well-posedness for (1.1) provided that  $H(u_0) \leq 1$ . The case where  $H(u_0) = 1$  is more subtle than the case where  $H(u_0) < 1$ . We also establish that well-posedness fails to hold on the set of initial data where  $H(u_0) > 1$ . Thus, we establish a complete trichotomy analogous to the energy critical cases of  $NLS_p(\mathbb{R}^d)$  in dimensions  $d \geq 3$ . Based on these results, we argue that (1.1) should be viewed as the energy critical NLS problem on  $\mathbb{R}^2$ . Using a new interaction Morawetz estimate, proved independently by Colliander-Grillakis-Tzirakis and Planchon-Vega [13, 31], the scattering was recently shown in [23] for subcritical solutions of (1.1) (with  $f(u) = u(e^{4\pi|u|^2} - 1 - 4\pi|u|^2)$ ). This problem remains open when  $H(u_0) = 1$  due to

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<sup>1</sup>Global well-posedness for the defocusing energy supercritical  $NLS_p(\mathbb{R}^d)$  with  $s_c = \frac{d}{2} - \frac{2}{p-1} > 1$  is an outstanding open problem.

the lack of uniform global estimates of the nonlinear term and the infinite speed of propagation.

*Remark 1.1.* The critical threshold for local and global well-posedness of (1.1) is expressed in terms of the size of  $H(u_0)$ . In contrast, the critical threshold for energy critical (1.5) is expressed in terms of  $\|u_0\|_{H^1}$ . Positive results for data satisfying  $H(u_0) > 1$  and other conditions may give insights towards proving global well-posedness results for energy supercritical problems.

**1.4. Statements of results.** We begin by formally defining our notion of criticality and well-posedness for (1.1). We then give precise statements of the main results we obtain and make brief comments about the rest of the paper.

**Definition 1.2.** *The Cauchy problem associated to (1.1) and with initial data  $u_0 \in H^1(\mathbb{R}^2)$  is said to be subcritical if*

$$H(u_0) < 1.$$

*It is critical if  $H(u_0) = 1$  and supercritical if  $H(u_0) > 1$ .*

**Definition 1.3.** *We say that the Cauchy problem associated to (1.1) is locally well-posed in  $H^1(\mathbb{R}^2)$  if there exist  $E > 0$  and a time  $T = T(E) > 0$  such that for every  $u_0 \in B_E := \{ u_0 \in H^1(\mathbb{R}^2); \|\nabla u_0\|_{L^2} < E \}$  there exists a unique (distributional) solution  $u : [-T, T] \times \mathbb{R}^2 \rightarrow \mathbb{C}$  to (1.1) which is in the space  $\mathcal{C}([-T, T]; H_x^1)$ , and such that the solution map  $u_0 \mapsto u$  is uniformly continuous from  $B_E$  to  $\mathcal{C}([-T, T]; H_x^1)$ .*

*A priori*, one can estimate the nonlinear part of the energy (1.4) using the following Moser-Trudinger type inequalities (see [1], [28], [37]).

**Proposition 1.4 (Moser-Trudinger Inequality).**

*Let  $\alpha \in [0, 4\pi)$ . A constant  $c_\alpha$  exists such that*

$$(1.8) \quad \|\exp(\alpha|u|^2) - 1\|_{L^1(\mathbb{R}^2)} \leq c_\alpha \|u\|_{L^2(\mathbb{R}^2)}^2$$

*for all  $u$  in  $H^1(\mathbb{R}^2)$  such that  $\|\nabla u\|_{L^2(\mathbb{R}^2)} \leq 1$ . Moreover, if  $\alpha \geq 4\pi$ , then (1.8) is false.*

*Remark 1.5.* We point out that  $\alpha = 4\pi$  becomes admissible in (1.8) if we require  $\|u\|_{H^1(\mathbb{R}^2)} \leq 1$  rather than  $\|\nabla u\|_{L^2(\mathbb{R}^2)} \leq 1$ . Precisely, we have

$$\sup_{\|u\|_{H^1} \leq 1} \|\exp(4\pi|u|^2) - 1\|_{L^1(\mathbb{R}^2)} < +\infty$$

and this is false for  $\alpha > 4\pi$ . See [32] for more details.

To establish an energy estimate, one has to consider the nonlinearity as a source term in (1.1), so we need to estimate it in the  $L_t^1(H_x^1)$  norm. To do so, we use (1.8) combined with the so-called Strichartz estimate.

**Proposition 1.6 (Strichartz estimates).**

*Let  $v_0$  be a function in  $H^1(\mathbb{R}^2)$  and  $F \in L^1(\mathbb{R}, H^1(\mathbb{R}^2))$ . Denote by  $v$  the solution of the inhomogeneous linear Schrödinger equation*

$$i\partial_t v + \Delta v = F$$

with initial data  $v(0, x) = v_0(x)$ .

Then, a constant  $C$  exists such that for any  $T > 0$  and any admissible couple of Strichartz exponents  $(q, r)$  i.e  $0 \leq \frac{2}{q} = 1 - \frac{2}{r} < 1$ , we have

$$(1.9) \quad \|v\|_{L^q([0,T], \mathcal{B}_{r,2}^1(\mathbb{R}^2))} \leq C [\|v_0\|_{H^1(\mathbb{R}^2)} + \|F\|_{L^1([0,T], H^1(\mathbb{R}^2))}].$$

In particular, note that  $(q, r) = (4, 4)$  is an admissible Strichartz couple and

$$(1.10) \quad \mathcal{B}_{4,2}^1(\mathbb{R}^2) \longrightarrow \mathcal{C}^{1/2}(\mathbb{R}^2).$$

Recall that, for  $1 \leq p, q \leq \infty$  and  $s \in \mathbb{R}$ , the (inhomogeneous) Besov norm  $\|\cdot\|_{\mathcal{B}_{p,q}^s(\mathbb{R}^2)}$  is defined by

$$\|u\|_{\mathcal{B}_{p,q}^s(\mathbb{R}^2)} := \left( \sum_{j \geq -1} 2^{jq_s} \|\Delta_j u\|_{L^p}^q \right)^{\frac{1}{q}}$$

with the usual modification when  $q = \infty$ .  $(\Delta_j)$  is a (inhomogeneous) dyadic partition of unity.

*Remark 1.7.*

- The homogeneous Besov norm is defined in the same manner using a homogeneous dyadic partition of unity  $(\dot{\Delta}_j)_{j \in \mathbb{Z}}$ .
- The connection between Besov spaces and the usual Sobolev and Hölder spaces is given by the following relations

$$\mathcal{B}_{2,2}^s(\mathbb{R}^2) = H^s(\mathbb{R}^2), \quad \mathcal{B}_{\infty,\infty}^s(\mathbb{R}^2) = \mathcal{C}^s(\mathbb{R}^2).$$

We recall without proof the following properties of Besov spaces (see [34], [35] and [36]).

**Theorem 1.8 (Embedding result).**

*The following injection holds*

$$\mathcal{B}_{p,q}^s(\mathbb{R}^2) \hookrightarrow \mathcal{B}_{p_1,q_1}^{s_1}(\mathbb{R}^2)$$

where

$$\begin{cases} s - \frac{2}{p} = s_1 - \frac{2}{p_1}, \\ 1 \leq p \leq p_1 \leq \infty, \quad 1 \leq q \leq q_1 \leq \infty, \quad s, s_1 \in \mathbb{R}. \end{cases}$$

The following estimate is an  $L^\infty$  logarithmic inequality which enables us to establish the link between  $\|e^{4\pi|u|^2} - 1\|_{L_T^1(L^2(\mathbb{R}^2))}$  and dispersion properties of solutions of the linear Schrödinger equation.

**Proposition 1.9 (Log Estimate).**

Let  $\beta \in ]0, 1[$ . For any  $\lambda > \frac{1}{2\pi\beta}$  and any  $0 < \mu \leq 1$ , a constant  $C_\lambda > 0$  exists such that, for any function  $u \in H^1(\mathbb{R}^2) \cap \mathcal{C}^\beta(\mathbb{R}^2)$ , we have

$$(1.11) \quad \|u\|_{L^\infty}^2 \leq \lambda \|u\|_\mu^2 \log\left(C_\lambda + \frac{8^\beta \mu^{-\beta} \|u\|_{\mathcal{C}^\beta}}{\|u\|_\mu}\right),$$

where we set

$$(1.12) \quad \|u\|_{\mu}^2 := \|\nabla u\|_{L^2}^2 + \mu^2 \|u\|_{L^2}^2.$$

Recall that  $\mathcal{C}^{\beta}(\mathbb{R}^2)$  denotes the space of  $\beta$ -Hölder continuous functions endowed with the norm

$$\|u\|_{\mathcal{C}^{\beta}(\mathbb{R}^2)} := \|u\|_{L^{\infty}(\mathbb{R}^2)} + \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^{\beta}}.$$

We refer to [20] for the proof of this proposition and more details. We just point out that the condition  $\lambda > \frac{1}{2\pi\beta}$  in (1.11) is optimal.

Our first statement describes a local well-posedness result when the initial data is in the open unit ball of the homogeneous Sobolev space  $\dot{H}^1(\mathbb{R}^2)$ . The sign of the nonlinearity is irrelevant here. Consider the following equation :

$$(1.13) \quad i\partial_t u + \Delta u = \sigma f(u).$$

We have the following short time existence Theorem.

**Theorem 1.10.** *Let  $\sigma \in \{-1, +1\}$  and  $u_0 \in H^1(\mathbb{R}^2)$  such that  $\|\nabla u_0\|_{L^2(\mathbb{R}^2)} < 1$ . Then, there exists a time  $T > 0$  and a unique solution to the equation (1.13) in the space  $\mathcal{C}_T(H^1(\mathbb{R}^2))$  with initial data  $u_0$  .*

*Moreover,  $u \in L_T^4(\mathcal{C}^{1/2}(\mathbb{R}^2))$  and satisfies, for all  $0 \leq t < T$ ,  $M(u(t, \cdot)) = M(u_0)$  and  $H(u(t, \cdot)) = H(u_0)$ .*

The proof of this Theorem is similar to Theorem 1.8 in [19]. It is based on the combination of the three *a priori* estimates given by the above propositions. We derive the local well-posedness using a classical fixed point argument.

*Remark 1.11.* In [22] a weak well-posedness result was proved without any restriction on the size of the initial data. More precisely, it is shown that the solution map is only continuous, while Theorem 1.10 says that it is uniformly continuous when  $\|\nabla u_0\|_{L^2(\mathbb{R}^2)} < 1$ . Well-posedness results with merely continuous dependence upon the initial data have also been obtained for the KdV equation [24] using the completely integrable machinery and for the cubic NLS on the line [12], [27] using PDE methods.

*Remark 1.12.* In the defocusing case, the assumption  $H(u_0) \leq 1$  in particular implies that  $\|\nabla u_0\|_{L^2(\mathbb{R}^2)} < 1$ , and consequently we have the short-time existence of solutions in both subcritical and critical case. So it makes sense to investigate global existence in these cases.

As an immediate consequence of Theorem 1.10 we have the following global existence result.

**Theorem 1.13 (Subcritical case).**

*Assume that  $H(u_0) < 1$ ; then the defocusing problem (1.1) has a unique global solution  $u$  in the class*

$$\mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^2)).$$

*Moreover,  $u \in L_{loc}^4(\mathbb{R}, \mathcal{C}^{1/2}(\mathbb{R}^2))$  and satisfies the conservation laws (1.3) and (1.4).*

The reason behind Definition 1.2 is the following: If  $u$  denotes the solution given by Theorem 1.10, where  $T^* < \infty$  is the largest time of existence, then the conservation of the total energy gives us, in the subcritical setting, a uniform bound of  $\|\nabla u(t, \cdot)\|_{L^2(\mathbb{R}^2)}$  away from 1, and therefore the solution can be continued in time. In contrast, for the critical case, we lose this uniform control and the total energy can be concentrated in the  $\|\nabla u(t, \cdot)\|_{L^2(\mathbb{R}^2)}$  part. By using a localization result due to Nakanishi (see Lemma 6.2 in [29]), we show that such concentration cannot hold in the critical case and therefore we have the following theorem:

**Theorem 1.14 (Critical case).**

Assume that  $H(u_0) = 1$ ; then the problem (1.1) has a unique global solution  $u$  in the class

$$\mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^2)).$$

Moreover,  $u \in L^4_{loc}(\mathbb{R}, C^{1/2}(\mathbb{R}^2))$  and satisfies the conservation laws (1.3) and (1.4).

*Remark 1.15.* Recently in [23] the scattering was established in the subcritical case using a new estimate obtained independently in [13, 31].

When the initial data are more regular, we can easily prove that the solution remains regular. More precisely, we have the following theorem:

**Theorem 1.16.** Assume that  $u_0 \in H^s(\mathbb{R}^2)$  with  $s > 1$  and  $\|\nabla u_0\|_{L^2(\mathbb{R}^2)} < 1$ . Then, the solution  $u$  given in Theorem 1.10 is in the space  $\mathcal{C}_T(H^s(\mathbb{R}^2))$ .

*Remark 1.17.* In fact, the local well-posedness holds in  $H^s$  for  $s > 1$  without any assumption on the size of the initial data.

The last result in this paper concerns the supercritical case.

**Theorem 1.18 (The supercritical case).**

There exist sequences of initial data  $u_k(0)$  and  $v_k(0)$  bounded in  $H^1$  and satisfying

$$\liminf_{k \rightarrow \infty} H(u_k(0)) > 1, \quad \liminf_{k \rightarrow \infty} H(v_k(0)) > 1,$$

with

$$\lim_{k \rightarrow +\infty} \|u_k(0) - v_k(0)\|_{H^1} = 0,$$

but there exists a sequence of times  $t_k > 0$  with  $t_k \rightarrow 0$  and

$$\liminf_{k \rightarrow \infty} \|\nabla(u_k(t_k) - v_k(t_k))\|_{L^2} \gtrsim 1.$$

*Remark 1.19.* The sequences of initial data constructed in Theorem 1.18 do not have bounded Hamiltonians. Indeed, their potential parts are huge. Unlike for the Klein-Gordon where the speed of propagation is finite see [21, 22], we were unable to prove the above result for slightly supercritical data.

This class of two-dimensional problems with exponential growth nonlinearities has been studied, for small Cauchy data, by Nakamura and Ozawa in [30]. They proved global well-posedness and scattering.

**Notation.** Let  $T$  be a positive real number. We denote by  $\mathbf{X}(T)$  the Banach space defined by

$$\mathbf{X}(T) = \mathcal{C}_T(H^1(\mathbb{R}^2)) \cap L_T^4(\mathcal{C}^{1/2}(\mathbb{R}^2)),$$

and endowed with the norm

$$\|u\|_T := \sup_{t \in [0, T]} (\|u(t, \cdot)\|_{L^2} + \|\nabla u(t, \cdot)\|_{L^2}) + \|u\|_{L_T^4(\mathcal{C}^{1/2})}.$$

Here and below  $\mathcal{C}_T(X)$  denotes  $\mathcal{C}([0, T]; X)$  and  $L_T^p(X)$  denotes  $L^p([0, T]; X)$ .

If  $A$  and  $B$  are nonnegative quantities, we use  $A \lesssim B$  to denote  $A \leq CB$  for some positive universal constant  $C$ , and  $A \approx B$  to denote the estimate  $A \lesssim B \lesssim A$ . For every positive real number  $R$ ,  $B(R)$  denotes the ball in  $\mathbb{R}^2$  centered at the origin and with radius  $R$ .

## 2. LOCAL WELL-POSEDNESS

This section is devoted to the proof of Theorem 1.10 about local existence. We begin with the following Lemma which summarizes some of the properties of the exponential nonlinearity.

**Lemma 2.1** (Nonlinear Inhomogeneous Estimate). *Let  $f$  be the function given by (1.2),  $T > 0$  and  $0 \leq A < 1$ . There exists  $0 < \gamma = \gamma(A) < 3$  such that for any two functions  $U_1$  and  $U_2$  in  $\mathbf{X}(T)$  satisfying the following*

$$(2.1) \quad \sup_{t \in [0, T]} \|\nabla U_j(t, \cdot)\|_{L^2} \leq A,$$

we have

$$(2.2) \quad \begin{aligned} \|f(U_1) - f(U_2)\|_{L_T^1(H^1(\mathbb{R}^2))} &\lesssim \|U_1 - U_2\|_T \left\{ T^{\frac{3}{4}} \sum_{j=1,2} \|U_j\|_T^3 \right. \\ &\quad \left. + T^{\frac{3-\gamma}{4}} \sum_{j=1,2} \left\| \frac{U_j}{A} \right\|_T^\gamma \right\} \end{aligned}$$

*Proof of Lemma 2.1.* Let us identify  $f$  with the  $\mathcal{C}^\infty$  function defined on  $\mathbb{R}^2$  and denote by  $Df$  the  $\mathbb{R}^2$  derivative of the identified function. Then using the mean value theorem and the convexity of the exponential function, we derive the following properties:

$$|f(z_1) - f(z_2)| \lesssim |z_1 - z_2| \sum_{j=1,2} \left( e^{4\pi|z_j|^2} - 1 + |z_j|^2 e^{4\pi|z_j|^2} \right),$$

and

$$|(Df)(z_1) - (Df)(z_2)| \lesssim |z_1 - z_2| \sum_{j=1,2} \left( |z_j| e^{4\pi|z_j|^2} + |z_j|^3 e^{4\pi|z_j|^2} \right)$$



Therefore, for any positive real number  $\varepsilon$  there exists a positive constant  $C_\varepsilon$  such that

$$(2.3) \quad |f(z_1) - f(z_2)| \leq C_\varepsilon |z_1 - z_2| \left\{ e^{4\pi(1+\varepsilon)|z_1|^2} - 1 + e^{4\pi(1+\varepsilon)|z_2|^2} - 1 \right\},$$

and

$$(2.4) \quad |(Df)(z_1) - (Df)(z_2)| \leq C_\varepsilon |z_1 - z_2| \sum_{i=1,2} \left( |z_i| + e^{4\pi(1+\varepsilon)|z_i|^2} - 1 \right).$$

Now we estimate  $\|f(U_1) - f(U_2)\|_{L_T^1(L^2(\mathbb{R}^2))}$ . Applying the Hölder inequality and using (2.3) we infer

$$\|f(U_1)(t, \cdot) - f(U_2)(t, \cdot)\|_{L_T^1(L^2(\mathbb{R}^2))} \leq C_\varepsilon \|U_1 - U_2\|_{L_T^4(L^4)} \sum_{j=1,2} \|e^{4\pi(1+\varepsilon)|U_j(t, \cdot)|^2} - 1\|_{L_T^{4/3}(L^4)}.$$

Applying Hölder inequality again, we obtain

$$(2.5) \quad \left\| e^{4\pi(1+\varepsilon)|U_j(t, \cdot)|^2} - 1 \right\|_{L_T^{4/3}(L_x^4)} \leq \left\| e^{3\pi(1+\varepsilon)\|U_j(t, \cdot)\|_{L_x^\infty}^2} \right\|_{L_T^{4/3}} \left\| e^{4\pi(1+\varepsilon)|U_j(t, \cdot)|^2} - 1 \right\|_{L_T^\infty(L_x^1)}^{\frac{1}{4}}.$$

Thanks to the Moser-Trudinger inequality (1.8) and the Log estimate (1.11) we get

$$(2.6) \quad \left\| e^{4\pi(1+\varepsilon)|U_j(t, \cdot)|^2} - 1 \right\|_{L^1} \leq C_{4\pi(1+\varepsilon)A^2} \|U_j(t, \cdot)\|_{L^2}^2,$$

$$(2.7) \quad e^{3\pi(1+\varepsilon)\|U_j(t, \cdot)\|_{L_x^\infty}^2} \lesssim \left( e^3 + \frac{\|U_j(t, \cdot)\|_{C^{1/2}}}{A'} \right)^\gamma,$$

where we set

$$A'^2 := A^2 + \max_i \sup_{t \in [0, T]} \mu^2 \|U_j(t, \cdot)\|_{L^2}^2 \quad \text{and} \quad \gamma := 3\pi\lambda(1+\varepsilon)A'^2,$$

and  $0 < \mu \leq 1$  is chosen such that  $A' < 1$ . Remember that  $C_{4\pi(1+\varepsilon)A^2}$  is given by Proposition 1.4. It is important to note that estimate (2.6) is true as long as the parameter  $\varepsilon$  is such that  $(1+\varepsilon)A^2 < 1$ . Now, inserting this back into (2.5), and integrating with respect to time, we obtain

$$\|(f(U_1) - f(U_2))(t, \cdot)\|_{L_T^1(L^2(\mathbb{R}^2))} \lesssim \|U_1 - U_2\|_{L_T^4(L^4)} \sum_{j=1,2} \left\| e^3 + \frac{\|U_j\|_{C^{1/2}}}{A'} \right\|_{L_T^{\frac{4}{3}\gamma}}^\gamma \|U_j\|_{L_T^\infty(L^2)}^{1/2}.$$

Now we estimate  $\|f(U_1) - f(U_2)\|_{L^1([0, T], \dot{H}^1(\mathbb{R}^2))}$ . We write

$$\begin{aligned} D(f(U_1) - f(U_2)) &= [(Df)(U_1) - (Df)(U_2)]DU_1 + Df(U_2)D(U_1 - U_2) \\ &:= (I) + (II). \end{aligned}$$

To estimate (I) we use (2.4). Hence for any  $\varepsilon > 0$  we have

$$|(Df)(U_1) - (Df)(U_2)| \leq C_\varepsilon |U_1 - U_2| \sum_{j=1,2} \left( e^{4\pi(1+\varepsilon)|U_j|^2} - 1 + |U_j| \right),$$

and therefore

$$\begin{aligned}
|(I)| &\lesssim |U_1 - U_2| \|DU_1\| \sum_{j=1,2} |U_j| \\
&+ |U_1 - U_2| \|DU_1\| \sum_{j=1,2} \left( e^{4\pi(1+\varepsilon)|U_j|^2} - 1 \right).
\end{aligned}$$

Applying Hölder inequality we infer

$$\begin{aligned}
\|(I)\|_{L^2(\mathbb{R}^2)} &\lesssim \|U_1 - U_2\|_{L^8(\mathbb{R}^2)} \|DU_1\|_{L^4(\mathbb{R}^2)} \sum_{j=1,2} \|U_j\|_{L^8(\mathbb{R}^2)} \\
&+ \|U_1 - U_2\|_{L^{4(1+\frac{1}{\varepsilon})}(\mathbb{R}^2)} \|DU_1\|_{L^4(\mathbb{R}^2)} \sum_{j=1,2} \|e^{4\pi(1+\varepsilon)|U_j|^2} - 1\|_{L^{4(1+\varepsilon)}}.
\end{aligned}$$

Using (2.6) and integrating with respect to time we deduce that

$$\begin{aligned}
\|(I)\|_{L_T^1(L^2)} &\lesssim T^{\frac{3}{4}} \|U_1 - U_2\|_{L_T^\infty(L^8)} \|DU_1\|_{L^4([0,T] \times \mathbb{R}^2)} \sum_{j=1,2} \|U_j\|_{L_T^\infty(L^8)} \\
&+ \|U_1 - U_2\|_{L_T^\infty(L^{4(1+\frac{1}{\varepsilon})})} \|DU_1\|_{L^4([0,T] \times \mathbb{R}^2)} \sum_{j=1,2} \left\| e^3 + \frac{\|U_j(t, \cdot)\|_{L^{\frac{1}{2}}}^\gamma}{A'} \right\|_{L_T^{\frac{4}{3}\gamma}}.
\end{aligned}$$

To estimate the term  $(II)$ , we use (1.8) with  $U_1 = 0$ . So thanks to the Hölder inequality we get

$$\begin{aligned}
\|(II)\|_{L^2(\mathbb{R}^2)} &\lesssim \|D(U_1 - U_2)\|_{L^4(\mathbb{R}^2)} \|U_2\|_{L^8(\mathbb{R}^2)}^2 \\
&+ \|D(U_1 - U_2)\|_{L^4(\mathbb{R}^2)} \|U_2\|_{L^{4(1+\frac{1}{\varepsilon})}(\mathbb{R}^2)} \sum_{j=1,2} \|e^{4\pi(1+\varepsilon)|U_j|^2} - 1\|_{L^{4(1+\varepsilon)}}.
\end{aligned}$$

Then we proceed exactly as we did for term  $(I)$ .

Now since  $A < 1$ , we can choose the parameter  $\mu$  such that  $A' < 1$ . Then we chose  $\varepsilon > 0$  small enough and  $\lambda > \frac{1}{\pi}$  and close to  $\frac{1}{\pi}$  such that  $\gamma < 3$ . Applying Hölder inequality (with respect to time) in the above inequality and in (2.6), we deduce (2.2) as desired.  $\blacksquare$

*Proof of Theorem 1.10.* The proof is divided into two steps.

**First step: Local existence.**

Let  $v_0$  be the solution of the free Schrödinger equation with  $u_0$  as the Cauchy data. Namely,

$$\begin{aligned}
(2.8) \quad &i\partial_t v_0 + \Delta v_0 = 0 \\
&v_0(0, x) = u_0.
\end{aligned}$$

For any positive real numbers  $T$  and  $\delta$ , denote by  $\mathcal{E}_T(\delta)$  the closed ball in  $\mathbf{X}(T)$  of radius  $\delta$  and centered at the origin. On the ball  $\mathcal{E}_T(\delta)$ , define the map  $\Phi$  by

$$(2.9) \quad v \longmapsto \Phi(v) := \tilde{v},$$

where

$$(2.10) \quad i\partial_t \tilde{v} + \Delta \tilde{v} = (v + v_0)(e^{4\pi|v+v_0|^2} - 1), \quad \tilde{v}(0, x) = 0.$$

Now the problem is to show that, if  $\delta$  and  $T$  are small enough, the map  $\Phi$  is well defined from  $\mathcal{E}_T(\delta)$  into itself and it is a contraction.

In order to show that the map  $\Phi$  is well defined, we need to estimate the term  $\|(v + v_0)(e^{4\pi|v+v_0|^2} - 1)\|_{L_T^1(H^1)}$ .

Let  $U_1 := v + v_0$ . Obviously,  $U_1 \in \mathbf{X}(T)$ . Moreover, since

$$\|\nabla v_0(t, \cdot)\|_{L^2} = \|\nabla u_0\|_{L^2}$$

is conserved along time, and  $\|\nabla u_0\|_{L^2} < 1$ , then the hypothesis (2.1) of Lemma 2.1 is satisfied. Now taking  $U_2 = 0$ , applying (2.2) and choosing  $\delta$  and  $T$  small enough show that  $\Phi$  is well defined. We do similarly for the contraction.  $\blacksquare$

### Second step: Uniqueness in the energy space.

The uniqueness in the energy space is a straightforward consequence of the following lemma and Theorem 1.10. Note that uniqueness in  $\mathbf{X}(T)$  follows from the contraction argument. Here we are noting the stronger statement that uniqueness holds in a larger space.

**Lemma 2.2.** *Let  $\delta$  be a positive real number and  $u_0 \in H^1(\mathbb{R}^2)$  such that  $\|\nabla u_0\|_{L^2} < 1$ . If  $u \in \mathcal{C}([0, T], H^1(\mathbb{R}^2))$  is a solution of (1.1)-(1.2) on  $[0, T]$ , then there exists a time  $0 < T_\delta \leq T$  such that  $u ; \nabla u \in L^4([0, T_\delta], L^4(\mathbb{R}^2))$  and*

$$\|u\|_{L^4([0, T_\delta] \times \mathbb{R}^2)} + \|\nabla u\|_{L^4([0, T_\delta] \times \mathbb{R}^2)} \leq \delta.$$

*Proof.* Fix  $a > 1$  such that

$$(2.11) \quad a\|\nabla u_0\|_{L^2}^2 < 1.$$

Then choose  $\varepsilon > 0$  such that

$$(2.12) \quad (1 + \varepsilon)^2 a \|\nabla u_0\|_{L^2}^2 < 1 \quad \text{and} \quad 4 \frac{(1 + \varepsilon)^2}{\varepsilon} \frac{a}{a - 1} \varepsilon^2 \leq 1.$$

Denote by  $V := u - v_0$  with  $v_0 := e^{it\Delta} u_0$ . Note that  $V$  satisfies

$$i\partial_t V + \Delta V = -(V + v_0)(e^{4\pi|V+v_0|^2} - 1).$$

According to the Strichartz inequalities, to prove that  $V$  and  $\nabla V$  are in  $L_{t,x}^4$  it is sufficient to estimate  $\nabla^j \left[ (V + v_0)(e^{4\pi|V+v_0|^2} - 1) \right]$  in the dual Strichartz norm  $\|\cdot\|_{L^{\frac{4}{3}}}$  with  $j = 0, 1$ .

By continuity of  $t \rightarrow V(t, \cdot)$ , one can choose a time  $0 < T_1 \leq T$  such that

$$\sup_{[0, T_1]} \|V(t, \cdot)\|_{H^1} \leq \varepsilon.$$

Moreover, observe that

$$|V + v_0|^2 \leq a|v_0|^2 + \frac{a}{a-1}|V|^2,$$

$$e^{x+y} - 1 = (e^x - 1)(e^y - 1) + (e^x - 1) + (e^y - 1),$$

and

$$xe^x \leq \frac{e^{(1+\varepsilon)x} - 1}{\varepsilon}.$$

We will only estimate the term with derivative, the other case is easier.

$$\begin{aligned} |\nabla[(V + v_0)(e^{4\pi|V+v_0|^2} - 1)]| &\leq |\nabla(V + v_0)(e^{4\pi|V+v_0|^2} - 1)| \\ &\quad + |\nabla(V + v_0)|V + v_0|^2 e^{4\pi|V+v_0|^2}| \\ &\leq |\nabla(V + v_0)(e^{4\pi|V+v_0|^2} - 1)| \\ &\quad + |\nabla(V + v_0)(e^{4\pi(1+\varepsilon)|V+v_0|^2} - 1)|. \end{aligned}$$

Hence we only need to estimate  $\|\nabla(V + v_0)(e^{4\pi(1+\varepsilon)|V+v_0|^2} - 1)\|_{L_{t,x}^{\frac{4}{3}}}$ . Applying the Hölder inequality we obtain

$$\|\nabla(V + v_0)(e^{4\pi(1+\varepsilon)|V+v_0|^2} - 1)\|_{L_{t,x}^{\frac{4}{3}}} \leq \|\nabla(V + v_0)\|_{L_t^\infty L_x^2} \|e^{4\pi(1+\varepsilon)|V+v_0|^2} - 1\|_{L_t^{\frac{4}{3}} L_x^4}.$$

Using the above observations we need to estimate the following three terms

$$\mathcal{I}_1(t) := \|(e^{4\pi(1+\varepsilon)a|v_0|^2} - 1)(e^{4\pi(1+\varepsilon)\frac{a}{a-1}|V|^2} - 1)\|_{L_t^{\frac{4}{3}} L_x^4}$$

$$\mathcal{I}_2(t) := \|e^{4\pi(1+\varepsilon)a|v_0|^2} - 1\|_{L_t^{\frac{4}{3}} L_x^4},$$

and

$$\mathcal{I}_3(t) := \|e^{4\pi(1+\varepsilon)\frac{a}{a-1}|V|^2} - 1\|_{L_t^{\frac{4}{3}} L_x^4}.$$

Applying Hölder inequality we obtain

$$\mathcal{I}_1(t) \leq \|e^{4\pi(1+\varepsilon)a|v_0|^2} - 1\|_{L_t^{\frac{4}{3}} L_x^{4(1+\varepsilon)}} \|e^{4\pi(1+\varepsilon)\frac{a}{a-1}|V|^2} - 1\|_{L_t^\infty L_x^{4\frac{1+\varepsilon}{\varepsilon}}}.$$

Now the choice of the parameters  $\varepsilon$  and  $a$  satisfying (2.11)-(2.12) insures that

$$\|e^{4\pi(1+\varepsilon)\frac{a}{a-1}|V|^2} - 1\|_{L^\infty([0, T_1], L^{4\frac{1+\varepsilon}{\varepsilon}})} \leq C(\varepsilon, a).$$

Also

$$\begin{aligned} \|e^{4\pi(1+\varepsilon)a|v_0|^2} - 1\|_{L^{\frac{4}{3}}([0, T_1], L^{4(1+\varepsilon)})} &\leq C(\varepsilon, a) \|e^{\pi(1+\varepsilon)a|v_0(t, \cdot)|^2}\|_{L^\infty}^3 \|L^4[0, T_1]\| \\ &\leq C(\varepsilon, a, T_1). \end{aligned}$$

Note that  $\lim_{S \rightarrow 0} C(\varepsilon, a, S) = 0$ , hence choosing  $T_1$  small enough we derive the desired estimate. The other terms can be estimated in a similar way. We omit the details here. ■

### 3. GLOBAL WELL-POSEDNESS

In this section, we start with a remark about the time of local existence. Then we show that the solutions emerging from the subcritical regime in the energy space extend globally in time by a rather simple argument. The more difficult critical case is then treated with a nonconcentration argument.

*Remark 3.1.* In Theorem 1.10, the time of existence  $T$  depends on  $u_0$ . However, in the case  $\|\nabla u_0\|_{L^2(\mathbb{R}^2)}^2 < 1 - \eta$ , this time of existence depends only on  $\eta$  and  $\|u_0\|_{L^2(\mathbb{R}^2)}^2$ .

**3.1. Subcritical Case.** Recall that in the subcritical setting we have  $H(u_0) < 1$ . Since the assumption  $H(u_0) < 1$  particularly implies that

$$\|\nabla u_0\|_{L^2} < 1,$$

it follows that the equation (1.1) has a unique maximal solution  $u$  in the space  $\mathbf{X}(T^*)$  where  $0 < T^* \leq +\infty$  is the lifespan of  $u$ . We want to show that  $T^* = +\infty$  which means that our solution is global in time.

*Proof of Theorem 1.13.* Assume that  $T^* < +\infty$ , then by the conservation of the Hamiltonian (identity (1.4)), we deduce that

$$\sup_{t \in [0, T^*)} \|\nabla u(t, \cdot)\|_{L^2(\mathbb{R}^2)} \leq H(u_0) < 1.$$

Now, let  $0 < s < T^*$  and consider the following Cauchy problem

$$\begin{cases} i\partial_t v + \Delta v &= f(v) \\ v(s, x) &= u(s, x) \in H^1(\mathbb{R}^2). \end{cases}$$

A fixed point argument (as that used in the proof of Theorem 1.10) shows that there exists a nonnegative  $\tau$  and a unique solution  $v$  to our problem on the interval  $[s, s + \tau]$ . Notice that  $\tau$  does not depend on  $s$  (see Remark 3.1 above). Choosing  $s$  close to  $T^*$  such that  $T^* - s < \tau$  the solution  $u$  can be continued beyond the time  $T^*$  which is a contradiction. ■

**3.2. Critical Case.** Now, we consider the case when  $H(u_0) = 1$ , and we want to prove a global existence result as in the subcritical setting.

The situation here is more delicate than that in the subcritical setting; in fact the arguments used there do not apply here. Let us briefly explain the major difficulty. Since  $H(u_0) = 1$  and by the conservation identities (1.3) and (1.4), it is possible (at least formally) that a concentration phenomena occurs, namely

$$\limsup_{t \rightarrow T^*} \|\nabla u(t, \cdot)\|_{L^2} = 1$$

where  $u$  is the maximal solution and  $T^* < +\infty$  is the lifespan of  $u$ . In such a case, we can not apply the previous argument to continue the solution. The actual proof is based on proving that the concentration phenomenon does not happen.

Arguing by contradiction we claim the following.

**Proposition 3.2.** *Let  $u$  be the maximal solution of (1.1) defined on  $[0, T^*)$ , and assume that  $T^*$  is finite. Then*

$$(3.1) \quad \limsup_{t \rightarrow T^*} \|\nabla u(t)\|_{L^2(\mathbb{R}^2)} = 1,$$

and

$$(3.2) \quad \limsup_{t \rightarrow T^*} \|u(t)\|_{L^4(\mathbb{R}^2)} = 0.$$

*Proof.* Note that for all  $0 \leq t < T^*$  we have

$$\|\nabla u(t)\|_{L^2(\mathbb{R}^2)}^2 \leq H(u(t, \cdot)).$$

On the other hand, since the Hamiltonian is conserved, we have

$$\limsup_{t \rightarrow T^*} \|\nabla u(t)\|_{L^2(\mathbb{R}^2)} \leq 1.$$

Assume that

$$\limsup_{t \rightarrow T^*} \|\nabla u(t)\|_{L^2(\mathbb{R}^2)} = L < 1.$$

Then, a time  $t_0$  exists such that  $0 < t_0 < T^*$  and

$$t_0 < t < T^* \implies \|\nabla u(t)\|_{L^2(\mathbb{R}^2)} \leq \frac{L+1}{2}.$$

Take a time  $s$  such that  $t_0 < s < T^* < s + \tau$  where  $\tau$  depends only on  $\frac{1-L}{2}$ . Using the local existence theory, we can extend the solution  $u$  after the time  $T^*$  which is a contradiction. This concludes the proof of (3.1).

To establish (3.2), it is sufficient to note that

$$2\pi|u(t, x)|^4 \leq \frac{e^{4\pi|u(t, x)|^2} - 1}{4\pi} - |u(t, x)|^2$$

and then consider the Hamiltonian with (3.1). ■

To localize the concentration and get a contradiction, the proof in the case of the nonlinear Klein-Gordon equation was crucially based on the property of finite speed of propagation satisfied by the solutions (see [19]). Here that property breaks down. Instead, we use the following localization result due to Nakanishi (see Lemma 6.2 in [29]).

**Lemma 3.3.** *Let  $u$  be a solution of (1.1) on  $[0, T)$  with  $0 < T \leq +\infty$  and suppose that  $E := H(u_0) + M(u_0) < \infty$ . A constant  $C(E)$  exists such that, for any two positive real numbers  $R$  and  $R'$  and for any  $0 < t < T$ , the following holds:*

$$(3.3) \quad \int_{B(R+R')} |u(t, x)|^2 dx \geq \int_{B(R)} |u_0(x)|^2 dx - C(E) \frac{t}{R'}.$$

For the sake of completeness, we shall give the proof here.

*Proof of Lemma 2.6 [29].* Let  $d_R(x) := d(x, B(R))$  be the distance from  $x$  to the ball  $B(R)$ . Obviously we have  $|\nabla d_R(x)| \leq 1$ . Define the cut-off function

$$\xi(x) := h\left(1 - \frac{d_R(x)}{R'}\right)$$

where  $h$  is a smooth function such that  $h(\tau) = 1$  if  $\tau \geq 1$  and  $h(\tau) = 0$  if  $\tau \leq 0$ . Note that  $\xi$  satisfies

$$\xi(x) = 1 \quad \text{if } x \in B(R), \quad \xi(x) = 0 \quad \text{if } |x| \geq R+R' \quad \text{and} \quad \|\nabla \xi(x)\|_{L^\infty} \lesssim 1/R'.$$

Multiplying equation (1.1) by  $\xi^2 \bar{u}$ , integrating on  $\mathbb{R}^2$  and taking the imaginary part, we get the following identity

$$\partial_t \|\xi u\|_{L^2}^2 = 4 \operatorname{Im} \left( \int_{\mathbb{R}^2} \xi \nabla \xi u \nabla \bar{u} dx \right) \geq -\frac{C(E)}{R'}.$$

This completes the proof of the Lemma. ■

*Proof of Theorem 1.14.* The proof of Theorem 1.14 is now straightforward. Assuming that  $T^* < +\infty$  and applying Hölder inequality to the left hand side of (3.3), we infer

$$\int_{B(R)} |u_0(x)|^2 dx - C(E) \frac{t}{R'} \leq \sqrt{\pi} (R + R') \|u(t)\|_{L^4(\mathbb{R}^2)}^2.$$

Taking first the *limsup* as  $t$  goes to  $T^*$  and then  $R'$  to infinity we deduce that  $u_0$  should be zero which leads to a contradiction and therefore the proof is achieved. ■

## 4. INSTABILITY OF SUPERCRITICAL SOLUTIONS OF NLS

The aim of this section is to show that the Cauchy problem (1.1) is ill-posed for certain data satisfying  $H(u_0) > 1$ . A typical example of supercritical data is the function  $f_k$  defined by:

$$f_k(x) = \begin{cases} 0 & \text{if } |x| \geq 1, \\ -\frac{\log|x|}{\sqrt{k\pi}} & \text{if } e^{-k/2} \leq |x| \leq 1, \\ \sqrt{\frac{k}{4\pi}} & \text{if } |x| \leq e^{-k/2}. \end{cases}$$

These functions were introduced in [28] to show the optimality of the exponent  $4\pi$  in Trudinger-Moser inequality (see also [1]).

An easy computation shows that  $\|\nabla f_k\|_{L^2(\mathbb{R}^2)} = 1$ . Since the sequence of functions  $f_k$  is not smooth enough, we begin by regularizing it in a way that preserves its “shape” i.e. : Let  $\chi$  be a smooth function such that  $0 \leq \chi \leq 1$  and

$$\chi(\tau) = \begin{cases} 0 & \text{if } \tau \leq 3/2, \\ 1 & \text{if } \tau \geq 2. \end{cases}$$

For every integer  $k \geq 1$ , let  $\eta_k(x) = \chi(e^{k/2}|x|)\chi(e^{k/2}(1 - |x|))$  and  $\tilde{f}_k = \eta_k f_k$ . An easy computation show that, for all  $j \geq 0$ , we have

$$\|\eta_k^{(j)}\|_{L^\infty} \leq e^{jk/2}.$$

For any nonnegative  $\alpha$  and  $A > 0$ , denote by

$$g_{\alpha,A,k}(y) := \left(1 + \frac{\alpha}{k}\right) \tilde{f}_k(y) \varphi\left(\frac{y}{\nu_k(A)}\right),$$

where  $\varphi$  is a cut-off function such that

$$\text{supp}(\varphi) \subset B(2), \quad \varphi = 1 \quad \text{on} \quad B(1), \quad 0 \leq \varphi \leq 1,$$

and the following choice of the scale  $\nu$

$$\nu_k(A) = \exp\left(-\frac{\sqrt{k}}{A}\right)$$

The cut-off function  $\varphi$  is made to insure that the rescaled  $g_{\alpha,A,k}(\nu_k(A)x)$  has a finite  $L^2$  norm. Now, let  $u$  solve the Cauchy problem

$$(4.1) \quad \begin{cases} i\partial_t u + \Delta_x u = f(u) \\ u(0, x) = g_{\alpha,A,k}(\nu_k(A)x). \end{cases}$$

Define  $v(t, \nu_k(A)x) = u(t, x)$ . Then  $v$  satisfies



$$(4.2) \quad \begin{cases} i\partial_t v + \nu_k(A)^2 \Delta_y v = f(v) \\ v(0, y) = g_{\alpha, A, k}(y) \end{cases}$$

For the sake of clarity, we omit the dependence of  $u$  and  $v$  upon the parameters  $\alpha$ ,  $k$  and  $A$ . We begin by showing that the initial data is supercritical.

**Lemma 4.1.** *There exists a positive constant  $C_1$  such that for every  $A > 0$ , we have*

$$\liminf_{k \rightarrow \infty} H(g_{\alpha, A, k}(\nu_k(A) \cdot)) \geq 1 + \frac{C_1}{\pi A^2}.$$

*Proof of Lemma 4.1.* For simplicity, we shall denote  $g_{\alpha, A, k}$  by  $g$  and  $\nu_k(A)$  by  $\nu$ . Recall that, by definition, we have

$$g(y) = 0 \quad \text{if } |y| \geq 2\nu \quad \text{or} \quad |y| \leq \frac{3}{2}e^{-k/2}$$

and

$$g(y) = \left(1 + \frac{\alpha}{k}\right) f_k(y) \quad \text{if } |y| \leq \nu \quad \text{and} \quad 2e^{-k/2} \leq |y| \leq 1 - 2e^{-k/2}$$

Remark that

$$H(g) \geq \|\nabla g\|_{L^2}^2 \geq (I) + (II),$$

where

$$(I) = \|\nabla g\|_{L^2(2e^{-k/2} \leq |y| \leq \nu)}^2$$

$$(II) = \|\nabla g\|_{L^2(\nu \leq |y| \leq 2\nu)}^2.$$

On the set  $\{2e^{-k/2} \leq |y| \leq \nu\}$  we have  $g(y) = -(1 + \alpha/k) \frac{\log |y|}{\sqrt{k\pi}}$  and thus

$$(4.3) \quad (I) = 1 - \frac{2}{A\sqrt{k}} + \frac{2(\alpha - \log 2)}{k} - \frac{4\alpha}{Ak^{3/2}} - \frac{4\alpha \log 2}{k^2}.$$

For the second term, we write

$$(4.4) \quad (II) = (a) + (b) + (c)$$

where

$$(a) = \left(1 + \frac{\alpha}{k}\right)^2 \int |\nabla f_k(y)|^2 |\varphi(\frac{y}{\nu})|^2 dy$$

$$(b) = \left(1 + \frac{\alpha}{k}\right)^2 \nu^{-2} \int |f_k(y)|^2 |\nabla \varphi(y)|^2 dy$$

$$(c) = 2\left(1 + \frac{\alpha}{k}\right)^2 \nu^{-1} \int f_k(y) \varphi(\frac{y}{\nu}) \nabla f_k(y) \cdot \nabla \varphi(\frac{y}{\nu}) dy$$

Clearly

$$(4.5) \quad (a) = \frac{2}{k} \left(1 + \frac{\alpha}{k}\right)^2 \left( \int_1^2 \frac{|\varphi(r)|^2}{r} dr \right),$$

and

$$(b) = \frac{1}{\pi k \nu^2} \left(1 + \frac{\alpha}{k}\right)^2 \int_{\nu \leq |y| \leq 2\nu} \log^2 |y| |\nabla \varphi(\frac{y}{\nu})|^2 dy.$$

But since, for  $k$  large,  $\left(\log 2 - \frac{\sqrt{k}}{A}\right)^2 \leq \log^2 |y| \leq \frac{k}{A^2}$ , we deduce that

$$\frac{1}{\pi k} \left(1 + \frac{\alpha}{k}\right)^2 \left(\log 2 - \frac{\sqrt{k}}{A}\right)^2 \left(\int_{1 \leq |z| \leq 2} |\nabla \varphi(z)|^2 dz\right) \leq (b) \leq \frac{C}{\pi A^2} (1 + \alpha/k)^2,$$

and therefore,

$$(4.6) \quad \left(1 + \frac{\alpha}{k}\right)^2 \left(\frac{C_1}{\pi A^2} - \frac{2 \log 2 C_1}{\pi A \sqrt{k}} + \frac{C_1 \log^2 2}{\pi k}\right) \leq (b) \leq \frac{C}{\pi A^2} (1 + \alpha/k)^2.$$

The constant  $C_1 = \|\nabla \varphi\|_{L^2}^2$ . For the last term, we simply write

$$(4.7) \quad \begin{aligned} (c) &= \frac{2}{\pi k} \left(1 + \frac{\alpha}{k}\right)^2 \int_{1 \leq |z| \leq 2} (\log \nu + \log |z|) \varphi(z) \frac{z \cdot \nabla \varphi(z)}{|z|^2} dz \\ &= \left(1 + \frac{\alpha}{k}\right)^2 \left(\frac{a}{\pi k} - \frac{b}{\pi A \sqrt{k}}\right), \end{aligned}$$

where the constants  $a$  and  $b$  are given by

$$a = 2 \int_{1 \leq |z| \leq 2} \log |z| \varphi(z) \frac{z \cdot \nabla \varphi(z)}{|z|^2} dz \quad \text{and} \quad b = 2 \int_{1 \leq |z| \leq 2} \varphi(z) \frac{z \cdot \nabla \varphi(z)}{|z|^2} dz$$

Finally, (4.3), (4.4) together with (4.5), (4.6) and (4.7) imply that for every  $A > 0$ ,

$$1 + \frac{C_1}{\pi A^2} \leq \liminf_{k \rightarrow \infty} H(g). \quad \blacksquare$$

The main result of this section reads.

**Theorem 4.2.** *Let  $\alpha > 0$  and  $A > 0$  be real numbers, and*

$$u_k(0, x) = g_{\alpha, A, k}(\nu_k(A)x),$$

$$v_k(0, x) = g_{0, A, k}(\nu_k(A)x).$$

*Denote by  $u_k$  and  $v_k$  the associated solutions of (1.1). Then, there exists a sequence  $t_k \rightarrow 0^+$  such that*

$$(4.8) \quad \liminf_{k \rightarrow \infty} \|\nabla(u_k - v_k)(t_k, \cdot)\|_{L^2(\mathbb{R}^2)} \gtrsim 1.$$

A general strategy to prove such instability result is to analyze the associated ordinary differential equation (see for instance, [10, 11]). More precisely, let  $\Phi$  solve

$$(4.9) \quad \begin{cases} i \partial_t \Phi(t, y) = f(\Phi(t, y)), \\ \Phi(0, y) = g_{\alpha, A, k}(y). \end{cases}$$

The problem (4.9) has an explicit solution given by:

$$\begin{aligned}\Phi_0^{(\alpha,A,k)}(t,y) &= g_{\alpha,A,k}(y) \exp\left(-it(e^{4\pi g_{\alpha,A,k}(y)^2} - 1)\right) \\ &:= g_{\alpha,A,k}(y) \exp(-itK(g_{\alpha,A,k}(y)))\end{aligned}$$

where  $K(z) = e^{4\pi|z|^2} - 1$ .

In the case of a power type nonlinearity, the common element in all arguments is a quantitative analysis of the NLS equation in the *small dispersion* limit

$$i\partial_t\Phi + \nu^2 \Delta\Phi = \sigma |\Phi|^{p-1}\Phi$$

where (the dispersion coefficient)  $\nu$  is small. Formally, as  $\nu \rightarrow 0$  this equation approaches the ODE

$$i\partial_t\Phi = \sigma |\Phi|^{p-1}\Phi$$

which has an explicit solution (see [10, 11] for more details). This fact and the invariance of equations of the type (1.5) under the scaling  $\Phi \mapsto \Phi^\lambda$  defined by

$$\Phi^\lambda(t,x) := \lambda^{2/(1-p)} \Phi(\lambda^{-2}t, \lambda^{-1}x)$$

play a crucial role in the ill-posedness results obtained in [10, 11] to make the decoherence happen during the approximation.

Unfortunately, no scaling leaves our equation invariant and this seems to be the major difficulty since it forces us to suitably construct the initial data in Theorem 4.2. Our solution to this difficulty (and others) proceeds in the context of energy and Strichartz estimates for the following equation

$$(4.10) \quad i\partial_t\Phi + \nu^2 \Delta\Phi = f(\Phi)$$

It turns out that given the scale  $\nu_k(A)$ , then for times close to  $\frac{e^{-k}}{\sqrt{k}}$ , equation (4.10) approaches the associated ODE (4.9).

*Proof of Theorem 4.2.* The proof is divided into two steps.

**First Step: "Decoherence"**

The key Lemma is the following.

**Lemma 4.3.** *Let  $\mathcal{C}_k$  denote the ring  $\{2e^{-k/2} \leq |y| \leq 3e^{-k/2}\}$  and  $t_k^\varepsilon = \varepsilon \frac{e^{-k}}{\sqrt{k}}$ . Then, we have*

$$(4.11) \quad \varepsilon e^{2\alpha} e^{-C\frac{\alpha}{k}} e^{-C\frac{\alpha^2}{k^2}} \lesssim \|\nabla\Phi_0^{\alpha,A,k}(t_k^\varepsilon)\|_{L^2(\mathcal{C}_k)} \lesssim \left(1 + \frac{\alpha}{k}\right)^3 \left(\frac{1}{k} + \varepsilon e^{2\alpha} e^{C\frac{\alpha^2}{k}}\right)$$

where  $C$  stands for an absolute positive constant which may change from term to term.

*Proof of Lemma 4.3.* Write  $\Phi_0^\alpha$  for  $\Phi_0^{\alpha,A,k}$ , then

$$\|\nabla\Phi_0^\alpha(t)\|_{L^2(\mathcal{C}_k)}^2 = \|\nabla g\|_{L^2(\mathcal{C}_k)}^2 + 64\pi^2 t^2 \|g^2 e^{4\pi g^2} \nabla g\|_{L^2(\mathcal{C}_k)}^2$$

In view of the definition of  $\eta_k$  and  $\varphi$ , we get

$$\|\nabla\Phi_0^\alpha(t)\|_{L^2(\mathcal{C}_k)}^2 = \left(1 + \frac{\alpha}{k}\right)^2 \mathcal{I} + 64\pi^2 t^2 \left(1 + \frac{\alpha}{k}\right)^6 \mathcal{J}$$

where

$$\begin{aligned} \mathcal{I} &= \int_{\mathcal{C}_k} |\nabla f_k(y)|^2 dy = \frac{2}{k} \log\left(\frac{3}{2}\right) \lesssim \frac{1}{k} \\ \mathcal{J} &= \int_{\mathcal{C}_k} |f_k(y)|^4 |\nabla f_k(y)|^2 e^{8\pi(1+\frac{\alpha}{k})^2 f_k(y)^2} dy \\ &= \frac{2}{\pi^2 k^2} \int_{2e^{-k/2}}^{3e^{-k/2}} \exp\left(\frac{8}{k}\left(1 + \frac{\alpha}{k}\right)^2 \log^2 r\right) \log^4 r \frac{dr}{r} \end{aligned}$$

We conclude the proof by remarking that, for  $2e^{-k/2} \leq r \leq 3e^{-k/2}$ , we have

$$k^4 e^{k/2} e^{4\alpha} e^{2k} e^{-C\frac{\alpha}{k}} e^{-C\frac{\alpha^2}{k^2}} \lesssim \exp\left(\frac{8}{k}\left(1 + \frac{\alpha}{k}\right)^2 \log^2 r\right) \frac{\log^4 r}{r} \lesssim k^4 e^{k/2} e^{4\alpha} e^{2k} e^{C\frac{\alpha}{k}}$$

■

**Corollary 4.4.** *Let  $\alpha > 0$  be a real number. Then,*

$$(4.12) \quad \liminf_{k \rightarrow \infty} \left\| \nabla \left( \Phi_0^{\alpha, A, k} - \Phi_0^{0, A, k} \right) (t_k^\varepsilon) \right\|_{L^2(\mathbb{R}^2)} \gtrsim \varepsilon (e^{2\alpha} - 1).$$

*Proof.* In view of the previous lemma, we have

$$\begin{aligned} \left\| \nabla (\Phi_0^\alpha - \Phi_0^0) (t_k^\varepsilon) \right\|_{L^2(\mathbb{R}^2)} &\geq \left\| \nabla (\Phi_0^\alpha - \Phi_0^0) (t_k^\varepsilon) \right\|_{L^2(\mathcal{C}_k)} \\ &\geq \left\| \nabla \Phi_0^\alpha (t_k^\varepsilon) \right\|_{L^2(\mathcal{C}_k)} - \left\| \nabla \Phi_0^0 (t_k^\varepsilon) \right\|_{L^2(\mathcal{C}_k)} \\ &\gtrsim \varepsilon e^{2\alpha} e^{-C\frac{\alpha}{k}} e^{-C\frac{\alpha^2}{k^2}} - \left( \frac{1}{k} + \varepsilon \right) \end{aligned}$$

and the conclusion follows. ■

**Second Step: Approximation**

The end of the proof of Theorem 4.2 lies in the following technical lemmas.

**Lemma 4.5.** *The solution  $\Phi_0^{\alpha, A, k}$  of (4.9) satisfies*

$$(4.13) \quad \left\| \nabla^3 \Phi_0^{\alpha, A, k} (t) \right\|_{L^2} \lesssim \frac{e^k}{\sqrt{k}} \left(1 + tk^{1/3} e^k\right)^3$$

*Proof of Lemma 4.5.*

Write  $\Phi_0$  for  $\Phi_0^{\alpha,A,k}$  and  $g$  for  $g_{\alpha,A,k}$ . Clearly,

$$\begin{aligned}\nabla\Phi_0 &= \left(\nabla g - itgK'(g)\nabla g\right)e^{-itK(g)} := g_1e^{-itK(g)}, \\ \nabla^2\Phi_0 &= \left(\nabla g_1 - itg_1K'(g)\nabla g\right)e^{-itK(g)} := g_2e^{-itK(g)}, \\ \nabla^3\Phi_0 &= \left(\nabla g_2 - itg_2K'(g)\nabla g\right)e^{-itK(g)} := g_3e^{-itK(g)},\end{aligned}$$

so

$$\begin{aligned}\|\nabla^3\Phi_0\|_{L^2} &\lesssim \|\nabla g_2\|_{L^2} + t\|g_2K'(g)\nabla g\|_{L^2}, \\ &\lesssim \|\nabla^3g\|_{L^2} + tA_1 + t^2A_2 + t^3A_3,\end{aligned}$$

where

$$\begin{aligned}A_1 &= \|K'(g)\nabla^2g\nabla g\|_{L^2} + \|K''(g)(\nabla g)^3\|_{L^2} \\ &\quad + \|gK'''(g)(\nabla g)^3\|_{L^2} + \|gK''(g)\nabla^2g\nabla g\|_{L^2} + \|gK'(g)\nabla^3g\|_{L^2}, \\ A_2 &= \|(K'(g))^2(\nabla g)^3\|_{L^2} + \|gK'(g)K''(g)(\nabla g)^3\|_{L^2} + \|g(K'(g))^2\nabla^2g\nabla g\|_{L^2}, \\ A_3 &= \|g(K'(g))^3(\nabla g)^3\|_{L^2}.\end{aligned}$$

Now,

$$\begin{aligned}\|\nabla^3g\|_{L^2}^2 &\lesssim \int_{\frac{3}{2}e^{-k/2} \leq |y| \leq 2\nu_k(A)} |\nabla^3f_k|^2 dy + \text{l.o.t.}, \\ &\lesssim \frac{1}{k} \int_{\frac{3}{2}e^{-k/2}}^{2\nu_k(A)} \frac{dr}{r^5} + \text{l.o.t.}, \\ &\lesssim \frac{e^{2k}}{k}.\end{aligned}$$

On the other hand

$$\begin{aligned}\|g(K'(g))^3(\nabla g)^3\|_{L^2}^2 &\lesssim \int |g|^8 e^{24\pi g^2} |\nabla g|^6 dy \\ &\lesssim \frac{1}{k^7} \int_{\frac{3}{2}e^{-k/2}}^{2\nu_k(A)} \log^8 r e^{\frac{24}{k}(1+\frac{\alpha}{k})^2 \log^2 r} \frac{dr}{r^5} + \text{l.o.t}\end{aligned}$$

■

The next lemma states the energy and Strichartz estimates for NLS with small dispersion coefficient  $\nu$ .

**Lemma 4.6.** *Let  $v_0$  be a function in  $H^1(\mathbb{R}^2)$  and  $F \in L^1(\mathbb{R}, H^1(\mathbb{R}^2))$ . Denote by  $v$  the solution of the inhomogeneous linear Schrödinger equation*

$$i\partial_t v + \nu^2 \Delta_y v = F(t, y)$$

*with initial data  $u(0, y) = v_0(y)$ .*

*Then, a constant  $C$  exists such that for any  $T > 0$ , we have*

$$(4.14) \quad \begin{aligned} \|\nabla v\|_{L_T^\infty(L^2)} + \frac{1}{\nu} \|v\|_{L_T^\infty(L^2)} + \nu^{1/2} \|v\|_{L_T^4(\dot{C}^{1/2})} &\lesssim \|\nabla v_0\|_{L^2} + \frac{1}{\nu} \|v_0\|_{L^2} \\ &+ \|\nabla F\|_{L_T^1(L^2)} + \frac{1}{\nu} \|F\|_{L_T^1(L^2)}. \end{aligned}$$

This lemma can be obtained from the standard Strichartz estimates through an obvious scaling. It can be seen as a semiclassical Strichartz estimate which permits an extension of the approximation time. Also, this lemma plays a role in the NLS analysis that is played by finite propagation speed in the corresponding NLW arguments. Now we are ready to end the proof of Theorem 4.2. For this purpose, denote (for simplicity) by  $w := \Phi - \Phi_0$  where  $\Phi_0$  is given by (4.10) and  $\Phi$  solves the problem

$$(4.15) \quad \begin{cases} i\partial_t \Phi(t, y) + \nu^2 \Delta_y \Phi(t, y) = f(\Phi(t, y)), \\ \Phi(0, y) = g(y). \end{cases}$$

Set

$$\begin{aligned} M_0(w, t) &\stackrel{\text{def}}{=} \|w\|_{L^\infty([0, t]; L^2)} \\ M_1(w, t) &\stackrel{\text{def}}{=} \|\nabla w\|_{L^\infty([0, t]; L^2)} + \frac{1}{\nu} \|w\|_{L^\infty([0, t]; L^2)} + \nu^{1/2} \|w\|_{L^4([0, t]; \dot{C}^{1/2})} \end{aligned}$$

We will prove the following result.

**Lemma 4.7.** *For  $t_k^\varepsilon \approx \varepsilon \frac{\varepsilon^{-k}}{\sqrt{k}}$  and  $k$  large, we have*

$$(4.16) \quad M_0(w, t_k^\varepsilon) \lesssim e^{-k/2} \nu^{3/2}.$$

$$(4.17) \quad M_1(w, t_k^\varepsilon) \lesssim \nu^{1/2}.$$

*Proof of Lemma 4.7.* Since  $w$  solves

$$i\partial_t w + \nu^2 \Delta_y w = f(\Phi_0 + w) - f(\Phi_0) - \nu^2 \Delta_y \Phi_0, \quad w(0, y) = 0,$$

then using the  $L^2$  energy estimate, we have

$$M_0(w, t) \lesssim \nu I_2(t) + \nu I_4(t),$$

where we set

$$I_2(t) : = \frac{1}{\nu} \|f(\Phi_0 + w) - f(\Phi_0)\|_{L^1([0, t]; L^2)},$$

$$I_4(t) : = \nu \|\nabla^2 \Phi_0\|_{L^1([0, t]; L^2)}.$$

Note that we have the following

$$\begin{aligned} \|(f(\Phi_0 + w) - f(\Phi_0))(t, \cdot)\|_{L^2} &\lesssim \|w(t, \cdot)\|_{L^2} \left( \|(\Phi_0 + w)(t, \cdot)\|_{L^\infty}^2 e^{4\pi \|(\Phi_0 + w)(t, \cdot)\|_{L^\infty}^2} \right. \\ &\quad \left. + \|w(t, \cdot)\|_{L^\infty}^2 e^{4\pi \|w(t, \cdot)\|_{L^\infty}^2} \right). \end{aligned}$$

Integrating in time we have

$$\begin{aligned} \|(f(\Phi_0 + w) - f(\Phi_0))(t, \cdot)\|_{L^1([0, T], L^2)} &\lesssim \int_0^t M_0(w, s) \left( \|(\Phi_0 + w)(s, \cdot)\|_{L^\infty}^2 e^{4\pi \|(\Phi_0 + w)(s, \cdot)\|_{L^\infty}^2} \right. \\ &\quad \left. + \|w(s, \cdot)\|_{L^\infty}^2 e^{4\pi \|w(s, \cdot)\|_{L^\infty}^2} \right) ds. \end{aligned}$$

Using Lemma 4.5 and the following simple fact

$$(4.18) \quad \sup_{x \geq 0} \left( x^m e^{-\gamma x^2} \right) = \left( \frac{m}{2\gamma} \right)^{m/2} e^{-m/2}, \quad m \in \mathbb{N}, \quad \gamma > 0,$$

we deduce that

$$(4.19) \quad M_0(w, t) \lesssim h_0(t) + \int_0^t A_0(s) M_0(w, s) ds,$$

where we set

$$h_0(t) = \nu^2 e^{k/2} \int_0^t (1 + s k e^k)^2 ds \lesssim t \nu^2 e^{k/2} (1 + (t k e^k)^2),$$

$$A_0(s) = k e^{4\pi(1+1/k)\|(\Phi_0 + w)(s, \cdot)\|_{L^\infty}^2} + k e^{4\pi(1+1/k)\|w(s, \cdot)\|_{L^\infty}^2}.$$

Applying the logarithmic inequality (1.11) ( for  $\lambda = \frac{1}{\pi}$ ) and using the fact that  $M_1(w, t) \lesssim \nu^{1/2}$ , we obtain

$$A_0(s) \lesssim k e^k (C + \nu^{-1/2} \|w\|_{C^{1/2}})^{4\delta_1(k)} + k (C + \nu^{-1/2} \|w\|_{C^{1/2}})^{4\delta_2(k)}$$

where

$$\delta_1(k) = (1 + 1/k)(\nu + \nu^{1/2} \sqrt{k}) \quad \text{and} \quad \delta_2(k) = (1 + 1/k)\nu.$$

Now, using Hölder inequality in time we deduce that

$$\begin{aligned} \int_0^t A_0(s) ds &\lesssim k e^k t^{1-\delta_1(k)} \left( t^{1/4} + \nu^{-1/2} \|w\|_{L_t^4(C^{1/2})} \right)^{4\delta_1(k)} \\ &\quad + k t^{1-\delta_2(k)} \left( t^{1/4} + \nu^{-1/2} \|w\|_{L_t^4(C^{1/2})} \right)^{4\delta_2(k)} \\ &\lesssim k e^k t^{1-\delta_1(k)} \left( t^{1/4} + \frac{1}{\sqrt{\nu}} \right)^{4\delta_1(k)} + \nu k^{3/2} t^{1-\delta_2(k)} \left( t^{1/4} + \frac{1}{\sqrt{\nu}} \right)^{4\delta_2(k)} \end{aligned}$$

It is easy to see that for  $t \approx t_k^\varepsilon$ ,

$$\int_0^t A_0(s) ds \lesssim \varepsilon \sqrt{k}.$$

Hence, by Gronwall's lemma

$$\begin{aligned} M_0(w, t_k^\varepsilon) &\lesssim h_0(t_k^\varepsilon) \exp(C\varepsilon\sqrt{k}) \\ &\lesssim \nu^{3/2} e^{-k/2} \frac{\varepsilon}{\sqrt{k}} (1 + \varepsilon^2 k) \exp\left(C\varepsilon\sqrt{k} - \frac{\sqrt{k}}{2A}\right) \\ &\lesssim \nu^{3/2} e^{-k/2} \end{aligned}$$

provided that  $\varepsilon < \frac{c}{A}$ .

Similarly we proceed for  $M_1$ . According to Lemma 4.6, we have

$$M_1(w, t) \lesssim I_1(t) + I_2(t) + I_3(t) + I_4(t),$$

where in addition we set

$$I_1(t) := \|\nabla(f(\Phi_0 + w) - f(\Phi_0))\|_{L^1([0,t];L^2)},$$

and

$$I_3(t) := \nu^2 \|\nabla^3 \Phi_0\|_{L^1([0,t];L^2)}.$$

Note that

$$\begin{aligned} \|\nabla(f(\Phi_0 + w) - f(\Phi_0))\|_{L_x^2} &\lesssim \|w\|_{L_x^2} \|\nabla \Phi_0\|_{L_x^\infty} \left( \|\Phi_0\|_{L_x^\infty} e^{4\pi\|\Phi_0\|_{L_x^\infty}^2} \right. \\ &\quad + \|\Phi_0\|_{L_x^\infty}^2 e^{4\pi\|\Phi_0\|_{L_x^\infty}^2} + \|w\|_{L_x^\infty} e^{4\pi\|w\|_{L_x^\infty}^2} \\ &\quad \left. + \|w\|_{L_x^\infty}^3 e^{4\pi\|w\|_{L_x^\infty}^2} \right) \\ &\quad + \|\nabla w\|_{L_x^2} \|\Phi_0 + w\|_{L_x^\infty}^2 e^{4\pi\|\Phi_0 + w\|_{L_x^\infty}^2} \\ &\quad + \|\nabla w\|_{L_x^2} e^{4\pi\|\Phi_0 + w\|_{L_x^\infty}^2}. \end{aligned}$$

Arguing as before, we have

$$\begin{aligned} (4.20) \quad M_1(w, t) &\lesssim h_1(t) + \frac{1}{\nu} \int_0^t A_0(s) M_0(w, s) ds + \int_0^t A_1(s) M_0(w, s) ds \\ &\quad + \int_0^t A_0(s) M_1(w, s) ds \end{aligned}$$

with in addition

$$\begin{aligned} h_1(t) &= \nu \|\nabla^3 \Phi_0\|_{L_t^1(L_x^2)} \lesssim \frac{e^k}{\sqrt{k}} (1 + tk^{1/3} e^k)^3, \\ A_1(s) &= \sqrt{k} e^{k/2} (1 + s k e^k) (e^k + \sqrt{k} e^{4\pi(1+1/k)\|w\|_\infty^2}). \end{aligned}$$



Here we used the Poincaré inequality and (4.18). Now we return to  $M_1(w, t)$  for which we have to prove (4.17). Using Lemma 4.5, (4.16) and Logarithmic inequality, we get for  $t \approx t_k^\varepsilon$

$$h_1(t) \lesssim \nu \frac{\varepsilon}{k} \left( 1 + \frac{\varepsilon^3}{\sqrt{k}} \right),$$

$$\frac{1}{\nu} \int_0^t M_0(w, s) A_0(s) ds \lesssim e^{-k/2} \nu^{1/2} \varepsilon \sqrt{k},$$

$$\int_0^t M_0(w, s) A_1(s) ds \lesssim \nu^{3/2} \varepsilon k (1 + \varepsilon \sqrt{k}).$$

Gronwall's lemma yields

$$M_1(w, t) \lesssim \nu^{1/2} \left( \nu^{1/2} + \varepsilon \sqrt{k} e^{-k/2} + \nu \varepsilon k (1 + \varepsilon \sqrt{k}) \right) \exp(C\varepsilon \sqrt{k}) \lesssim \nu^{1/2}$$

provided that  $\varepsilon < \frac{c}{A}$ . This completes the proof of Lemma 4.7 ■

Finally, a comparison of (4.11) with the approximation bounds (4.16), (4.17) implies (4.8). This completes the proof of Theorem 4.2. ■

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