Problem set 2 - Due 03/05/2012
Functional analysis - spring 2012

1) Suppose A is a linear operator on a Hilbert space X such that $D(A) = X$ and $(x, Ay) = (Ax, y)$ for all $x, y \in X$. Show that $A \in B(X)$.

2) Let $\{D_k\}$ be a sequence of dense open subsets of a B-space X. Show that

$$D = \cap_{k=1}^{\infty} D_k$$

is dense in X.

3) Let X be a B-space and Y a normed vector space. Prove that if T_n is a sequence in $B(X, Y)$ such that $\lim T_n x$ exists in Y for each $x \in X$, then there exists a $T \in B(X, Y)$ such that $T_n x \to Tx$ for all $x \in X$.

4) If X and Y are Hilbert spaces, is $B(X, Y)$ a Hilbert space ?

5) Prove the equivalence between the closed graph theorem, the open mapping theorem and the inverse map theorem.

6) Use Zorn's Lemma to show that every Hilbert space has an orthonormal basis.

Extra problem:

7) Also, an application of Zorn's Lemma shows that every linear space X has a Hamel basis, i.e. a set of independent vectors $\{x_i,\, i \in I\} \subset X$ such that any $x \in X$ can be represented in a unique way as

$$x = \sum_{k=1}^{n} \alpha_k x_{i_k}$$

for some $\alpha_k \in K$, $i_k \in I$, $x_{i_k} \in X$ and $n = n(x)$.

All Hamel bases have the same cardinality and their common dimension defines the dimension of X: $\dim(X)$.

8) An orthonormal basis of a Hilbert space X is a Hamel basis if an only if X is finite dimensional.