

**Problem set 6 - Due 04/09/2012**  
**Functional analysis - spring 2012**

1) Let  $A$  and  $B$  be operators in  $B(X)$  that commute. Show that

$$r_\sigma(AB) \leq r_\sigma(A)r_\sigma(B), \quad r_\sigma(A+B) \leq r_\sigma(A) + r_\sigma(B).$$

Can you find examples of strict inequalities.  
Is the commutation hypothesis necessary ?

2) (please ignore this problem. As it is, it is trivial.)  $A \in B(X)$  and  $p(t)$  a polynomial. Show that if  $\Phi_A$  is not empty, then  $p(A)$  is a closed operator.

3) If  $A \in B(X)$ , show that  $z(z - A)^{-1} \rightarrow I$  as  $|z| \rightarrow \infty$ .

4) Give an example of a pair of bounded operators such that  $0 \in \sigma(AB)$  but  $0 \notin \sigma(BA)$

5) Let  $X$  and  $Y$  be B-spaces. Let  $A \in B(X, Y)$  and  $B \in B(Y, X)$ . Suppose  $\mu \neq 0$ . Then

- a)  $R(\mu + AB)$  is closed *iff*  $R(\mu + BA)$  is closed
- b)  $\dim N(\mu + AB) = \dim N(\mu + BA)$