Problem set 9 - Due 04/30/2012
Functional analysis - spring 2012

1) Let A be a closed linear operator on X such that $(A-\lambda)^{-1}$ is compact for some $\lambda \in \rho(A)$. Show that $\sigma_e(A)$ is empty.

2) LEt A be a closed operator on X such that $0 \in \rho(A)$. Show that if $\lambda \neq 0$ then $\lambda \in \sigma(A)$ iff $1/\lambda \in \sigma(A^{-1})$.

3) Let A be defined on l^2 by

$$A(x_1, x_2, ...) = (x_1, 2x_2, ..., nx_n)$$

where $D(A)$ consists of those $x \in l^2$ such that $\sum_{n=1}^{\infty} |nx_n|^2 < \infty$.

a) Is A closed ? Is it densely defined ? Does A' exist ?

b) What are $\sigma(A)$, Φ_A and $\sigma_e(A)$?

4) Suppose $a \in C(\mathbb{R})$ and let T_a be an unbounded operator on $L^2(\mathbb{R})$ with domain $D(T_a) = \{f \in L^2 \,|\, af \in L^2\}$ and $T_a f = af$. What is the spectrum of T_a.

Assume $b \in L^1(\mathbb{R})$ and $C_b(f) = b * f$ (the convolution of b and f). Prove that C_b is a bounded operator on $L^2(\mathbb{R})$. What is its spectrum.

5) Consider the operator $T = \frac{d}{dx}$ on $C[a,b]$. We see it as an unbounded operator with domain $D(T) = C^1[a,b]$. Also consider

$T_1 = \frac{d}{dx}$ with $D(T_1) = D(T) \cap \{u \,|\, u(a) = 0\}$

$T_2 = \frac{d}{dx}$ with $D(T_2) = D(T) \cap \{u \,|\, u(b) = 0\}$

$T_3 = \frac{d}{dx}$ with $D(T_2) = D(T) \cap \{u \,|\, u(a) = u(b) = 0\}$

Find the spectrum of the operators T, T_1, T_2 and T_3.