Remarks about the inviscid limit of the Navier-Stokes system.

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Abstract

In this paper we prove two results about the inviscid limit of the Navier-Stokes system. The first one concerns the convergence in H^s of a sequence of solutions to the Navier-Stokes system when the viscosity goes to zero and the initial data is in H^s. The second result deals with the best rate of convergence for vortex patch initial data in 2 and 3 dimensions. We present here a simple proof which also works in the $3D$ case. The $3D$ case is new.

1 The inviscid limit

The Navier-Stokes system is the basic mathematical model for viscous incompressible flows. In a bounded domain, it reads

$$\begin{cases} \partial_t u + u.\nabla u - \nu\Delta u + \nabla p = 0, \\ \operatorname{div}(u) = 0, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where u is the velocity, p is the pressure and ν is the kinematic viscosity. We can define a typical length scale L and a typical velocity U. The dimensionless

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parameter $Re = \frac{UL}{\nu}$ is very important to compare the properties of different flows. When Re is very large (ν very small), we expect that the Navier-Stokes system (NS_ν) behaves like the Euler system

$$\begin{cases} \partial_t u + u.\nabla u + \nabla p = 0, \\ \operatorname{div}(u) = 0, \\ u.n = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{2}$$

The zero-viscosity limit of the incompressible Navier-Stokes equation in a bounded domain, with Dirichlet boundary conditions, is one of the most challenging open problems in Fluid Mechanics (see [19] and the references therein). This is due to the formation of a boundary layer which appears because, we can not impose a Dirichlet boundary condition for the Euler equation. This boundary layer satisfies formally the Prandtl equations, which seem to be ill-posed in general.

In this paper we only deal with the inviscid limit in the whole space. All the results presented here can easily be extended to the periodic case.

2 Convergence in H^s.

The inviscid limit in the whole space case was performed by several authors, we can refer for instance to Swann [20] and Kato [16, 17] (see also Constantin [10]). Here, we would like to improve slightly the convergence stated in the previous works by proving the convergence in the H^s space as long as the solution of the Euler system exists. Indeed, in most of the previous results only convergence in $H^{s'}$ for $s' < s$ was proved. We also point out that in [17], Kato proved the convergence in H^s for a short time, by using a general theory about quasi-linear equations. So, we do not claim that theorem 2.1 is really new.

Take the Navier-Stokes system in the whole space $\mathbb{R}^d$

$$\partial_t u^n + \operatorname{div}(u^n \otimes u^n) - \nu_n \Delta u^n = -\nabla p^n \quad \text{in} \quad \mathbb{R}^d \tag{3}$$

$$\operatorname{div}(u^n) = 0 \quad \text{in} \quad \mathbb{R}^d \tag{4}$$

$$u^n(t=0) = u_0^n \quad \text{with} \quad \operatorname{div}(u_0^n) = 0 \tag{5}$$

where ν_n goes to 0 when n goes to infinity.

Theorem 2.1 *Let $s > d/2 + 1$, and $u_0^n \in H^s(\mathbb{R}^d)$ such that u_0^n goes to u_0 in $H^s(\mathbb{R}^d)$ when n goes to infinity. Let T^* be the time of existence and $u \in C_{loc}([0, T^*); H^s)$ be the solution of the Euler system*

$$\partial_t u + \mathrm{div}(u \otimes u) = -\nabla p \quad in \quad \mathbb{R}^d \tag{6}$$

$$\mathrm{div}(u) = 0 \quad in \quad \mathbb{R}^d \tag{7}$$

$$u(t = 0) = u_0 \quad with \quad \mathrm{div}(u_0) = 0. \tag{8}$$

Then, for all $0 < T_0 < T^*$, there exists $\nu_0 > 0$ such that for all $\nu_n \leq \nu_0$, the Navier-Stokes system (3 - 5) has a unique solution $u^n \in C([0, T_0]; H^s(\mathbb{R}^d))$. Moreover,

$$\|u^n - u\|_{L^\infty(0,T_0;H^s)} \to 0, \quad n \to \infty \tag{9}$$

$$\|(u^n - u)(t)\|_{H^{s-2}} \leq C(\nu_n t + \|u_0^n - u_0\|_{H^{s-2}}) \tag{10}$$

$$\|(u^n - u)(t)\|_{H^{s'}} \leq C((\nu_n t)^{(s-s')/2} + \|u_0^n - u_0\|_{H^{s'}}) \tag{11}$$

for all $0 \leq t \leq T_0$, $s - 2 \leq s' \leq s - 1$ and C depends only on u and T_0.

Remark 2.2 *1) The only relatively new part in theorem 2.1 is the convergence in H^s stated in (9) which holds for all $T_0 < T^*$.*

2) Interpolating between (10) and the uniform bound for w^n in $C([0, T]; H^s(\mathbb{R}^d))$, we deduce that u^n converges to u in $H^{s'}$ for any $s' < s$ and for $s-2 < s' < s$, we have

$$\|(u^n - u)(t)\|_{H^{s'}} \leq C(\nu_n t + \|u_0^n - u_0\|_{H^{s-2}})^{\frac{s-s'}{2}}. \tag{12}$$

for all $0 \leq t \leq T_0$.

Proof:

The proof of this theorem is based on a standard Grönwall inequality (see also [20, 16, 10]). Let us start by proving (10). First, we see that we can solve the Navier-Stokes system and Euler system in $C([0, T]; H^s(\mathbb{R}^d))$ on some time interval independent of $\nu_n \leq \nu_0$ with bounds which are independent of n. This is because there is no boundary. Moreover, $w^n = u^n - u$ satisfies

$$\partial_t w^n + u^n \nabla w^n + w^n.\nabla u - \nu_n \Delta w^n + \nu_n \Delta u + \nabla(p^n - p) = 0 \tag{13}$$

Then, we can write an energy estimate in H^{s-2} for $w^n = u^n - u$, namely

$$\begin{aligned} &\partial_t \|w^n\|^2_{H^{s-2}} + \nu_n \|\nabla w^n\|^2_{H^{s-2}} \\ &\qquad \leq \Big(C(\|u\|_{H^s} + \|w^n\|_{H^s})\|w^n\|_{H^{s-2}} + \nu_n \|\Delta u\|_{H^{s-2}}\Big)\|w^n\|_{H^{s-2}} \end{aligned} \tag{14}$$

and by Grönwall lemma, we can deduce that (10) holds for $T_0 = T$. It was proved in [10] that the convergence holds as long as we can solve the Euler system and hence we can take any T_0 such that $T_0 < T^*$ (see [10]). Notice that in [10], the regularity required is $s - 2 > d/2 + 1$. However, this is not necessary modulo the regularization argument which is used to prove the convergence in H^s.

To prove (11), we write an energy estimate in $H^{s'}$, $s - 2 \leq s' \leq s - 1$

$$\partial_t \|w^n\|_{H^{s'}}^2 + \nu_n \|\nabla w^n\|_{H^{s'}}^2 \\ \leq \Big(C(\|u\|_{H^s} + \|w^n\|_{H^s}) \|w^n\|_{H^{s'}}^2 + \nu_n \|\nabla u\|_{H^{s-1}} \|\nabla w^n\|_{H^{2s'-(s-1)}} \Big). \quad (15)$$

Then using an interpolation inequality and Holder inequality, we deduce that

$$\|\nabla w^n\|_{H^{2s'-s+1}} \leq C \|w^n\|_{H^{s'}}^{s-s'-1} \|\nabla w^n\|_{H^{s'}}^{2-(s-s')} \leq \frac{1}{C} \|\nabla w^n\|_{H^{s'}}^2 + C^2 \|w^n\|_{H^{s'}}^{2-\frac{2}{s-s'}}.$$

Hence,

$$\partial_t \|w^n\|_{H^{s'}}^2 \leq C \|w^n\|_{H^{s'}}^2 + C\nu_n \|w^n\|_{H^{s'}}^{2-\frac{2}{s-s'}} \quad (16)$$

and (11) follows.

Getting the convergence in H^s requires a regularization of the initial data. For all $\delta > 0$, we take u_0^δ such that $\|u_0^\delta\|_{H^s} \leq C\|u_0\|_{H^s}$, $\|u_0^\delta\|_{H^{s+1}} \leq \frac{C}{\delta}$, $\|u_0^\delta\|_{H^{s+2}} \leq \frac{C}{\delta^2}$ and for some s' such that $d/2 < s' < s - 1$, we have $\|u_0^\delta - u_0\|_{H^{s'}} \leq C\delta^{s-s'}$. Such a u_0^δ can be easily constructed by taking $u_0^\delta = \mathcal{F}^{-1}(1_{\{|\xi| \leq 1/\delta\}} \mathcal{F} u_0)$.

Let v^δ be the solution of the Euler system (6,7,8) with the initial data $v^\delta(t = 0) = u_0^\delta$. It is easy to see that v^δ exists on some time interval $[0, T]$, $T < T^*$ which only depends on $\|u_0\|_{H^s}$ and such that for $0 \leq t \leq T$, we have $\|v^\delta(t)\|_{H^s} \leq C$ and $\|v^\delta(t)\|_{H^{s+2}} \leq \frac{C}{\delta^2}$ uniformly in δ. Indeed, the energy estimates at the level H^s and H^{s+2} read

$$\partial_t \|v^\delta\|_{H^s}^2 \leq C \|v^\delta\|_{H^s}^3 \quad (17)$$

$$\partial_t \|v^\delta\|_{H^{s+2}}^2 \leq C \|v^\delta\|_{H^s} \|v^\delta\|_{H^{s+2}}^2 \quad (18)$$

from which the uniform estimates follow.

Then, setting $w^\delta = v^\delta - u$, we have

$$\partial_t w^\delta + w^\delta.\nabla v^\delta + u.\nabla w^\delta = -\nabla(p^\delta - p). \quad (19)$$

Taking the energy estimate in H^s yields

$$\partial_t \|w^\delta\|_{H^s}^2 \leq C(\|u\|_{H^s} + \|v^\delta\|_{H^s})\|w^\delta\|_{H^s}^2 + C\|v^\delta\|_{H^{s+1}}\|w^\delta\|_{H^s}\|w^\delta\|_{L^\infty}. \quad (20)$$

Then, we notice that on the time interval $[0,T]$, we have $\|v^\delta\|_{H^{s+1}} \leq \frac{C}{\delta}$. Moreover, taking the energy estimate at the regularity s', we get

$$\partial_t \|w^\delta\|_{H^{s'}}^2 \leq C(\|u\|_{H^{s'}} + \|v^\delta\|_{H^{s'+1}})\|w^\delta\|_{H^{s'}}^2 \quad (21)$$

and since $s'+1 < s$, we get easily that $\|w^\delta\|_{L^\infty(0,T;H^{s'})} \leq C\delta^{s-s'}$ and by Sobolev embedding, we have $\|w^\delta\|_{L^\infty(0,T;L^\infty)} \leq C\|w^\delta\|_{L^\infty(0,T;H^{s'})} \leq C\delta^{s-s'}$. Hence, (20) gives

$$\partial_t \|w^\delta\|_{H^s} \leq C(\|u\|_{H^s} + \|v^\delta\|_{H^s}))\|w^\delta\|_{H^s} + C\delta^{s-s'-1}. \quad (22)$$

Hence w^δ goes to zero in $L^\infty(0,T;H^s)$, namely v^δ goes to v in $L^\infty(0,T;H^s)$ when δ goes to zero and we have

$$\|v^\delta - u\|_{L^\infty(0,T;H^s)} \leq C(\|u_0^\delta - u_0\|_{H^s} + \delta^{s-s'-1}T) \quad (23)$$

Writing an energy estimate for $w^{n,\delta} = u^n - v^\delta$, we get (here we drop the n and δ)

$$\begin{aligned}\partial_t \|w\|_{H^s}^2 &+ \nu_n \|\nabla w\|_{H^s}^2 \\ &\leq C(\|w\|_{L^\infty}\|v^\delta\|_{H^{s+1}}\|w\|_{H^s} + (\|v^\delta\|_{H^s} + \|u^n\|_{H^s})\|w\|_{H^s}^2) + \\ &\qquad \nu_n \|v^\delta\|_{H^{s+2}}\|w\|_{H^s}. \quad (24)\end{aligned}$$

Hence, we get

$$\begin{aligned}\partial_t \|w\|_{H^s} \leq C\|u^n - u\|_{L^\infty}\|v^\delta\|_{H^{s+1}} + C\|v^\delta - u\|_{L^\infty}\|v^\delta\|_{H^{s+1}} + \\ \nu_n \|v^\delta\|_{H^{s+2}} + C(\|v^\delta\|_{H^s} + \|u^n\|_{H^s})\|w\|_{H^s}. \quad (25)\end{aligned}$$

Since u^n converges to u in H^{s-1}, we deduce that

$$\|u^n - u\|_{L^\infty(0,T;L^\infty)} \leq C\|u^n - u\|_{L^\infty(H^{s-1})} \leq C((\nu_n T)^{1/2} + \|u_0^n - u_0\|_{H^{s-1}}). \quad (26)$$

Taking $\delta = \delta_n$ such that δ_n, $\frac{\|u_0^n - u_0\|_{H^{s-1}}}{\delta_n}$ and $\frac{\nu_n}{\delta_n^2}$ go to zero when n goes to infinity, we deduce that

$$\partial_t \|w^{n,\delta}\|_{H^s} \leq C(\frac{(\nu T)^{1/2} + \|u_0^n - u_0\|_{H^{s-1}}}{\delta} + \delta^{s-s'-1} + \frac{\nu}{\delta^2} + \|w^{n,\delta}\|_{H^s}) \quad (27)$$

Hence, by Grönwall lemma, we deduce that $w^{n,\delta}$ goes to zero in $L^\infty(0,T;H^s)$ and that u^n goes to u in $L^\infty(0,T;H^s)$. Moreover,

$$\|u^n - u\|_{L^\infty(0,T;H^s)} \le CT\left(\frac{(\nu T)^{1/2} + \|u_0^n - u_0\|_{H^{s-1}}}{\delta} + \delta^{s-s'-1} + \frac{\nu}{\delta^2}\right) + C(\|u_0^n - u_0\|_{H^s} + \|u_0^\delta - u_0\|_{H^s} + \delta^{s-s'-1}T). \quad (28)$$

We notice here that the rate of convergence gets better if we have a better approximation of u_0 by u_0^δ. This will be studied in the next subsection.

Since, we have proved the convergence in H^s till the time T, we can iterate the previous argument. Indeed, taking T as a new initial time and noticing that $u^n(T)$ goes to $u(T)$ in H^s, we see that we can iterate the previous argument on some time interval $[T, T+T_1]$ where $T_1 = T_1(\|u(T)\|_{H^s})$ only depends on $\|u(T)\|_{H^s}$ and $T_1 \ge C/\|u(T)\|_{H^s}$. Then, we can construct a sequence of times T_k, $k \ge 1$ by this procedure. Now, it is clear that $T + T_1 + ... + T_k$ goes to T^* when k goes to infinity. Indeed, the time T_{k+1} goes to zero only if $\|u(T + T_1 + ... + T_k)\|_{H^s}$ goes to infinity, which means that $T + T_1 + ... + T_k$ goes to T^*. This iteration argument allows us to get the convergence on any time interval $[0, T_0]$, $T_0 < T^*$. □

Remark 2.3 *1) We notice that the time T^* is related to the existence time for the Euler system (6). If d = 2 it is known [22, 21] that the Euler system (6) has a global solution and hence one can take any time $T_0 < \infty$ in the above theorem.*

2) The idea of using a regularization of the initial data was also used by Beirão da Veiga [2, 3] to prove a similar result in the compressible-incompressible limit. It is also used to prove the continuity of the solution with respect to the initial data in hyperbolic equations (see for instance Bona and Smith [5]). In the inviscid limit, this idea was used by Constantin and Wu [12] to prove some estimates on the rate of convergence of the vorticity.

2.1 Rate of convergence in H^s

Take β such that $1 < \beta \le 2$ and $d/2 < s - \beta$ and for $0 \le \delta < \infty$, $T > 0$ we define $u_0^\delta = \mathcal{F}^{-1}(1_{\{|\xi| \le 1/\delta\}}\mathcal{F}u_0)$, $\varepsilon_T(\delta) = \|u_0^\delta - u_0\|_{H^s} + T\delta^{\beta-1}$, $f_T(\delta) = \delta\varepsilon_T(\delta)$ and $g_T(\delta) = \delta^2\varepsilon_T(\delta)$. We can see easily that for $T > 0$, f_T and g_T are increasing on $[0,\infty)$. We denote by f_T^{-1} and g_T^{-1} their inverse. From the proof of theorem 2.1, we can deduce the following corollary

Corollory 2.4 *Under the same hypotheses of theorem 2.1, we have the following rate of convergence*

$$\|(u^n-u)(t)\|_{H^s} \le C\frac{\nu t}{(g_t^{-1}(\nu t))^2}+C\frac{t((\nu t)^{\beta/2}+\|u_0^n-u_0\|_{H^{s-\beta}})}{f_t^{-1}(t((\nu t)^{\beta/2}+\|u_0^n-u_0\|_{H^{s-\beta}}))}+C\|u_0^n-u_0\|_{H^s} \tag{29}$$

for all $0 \le t \le T_0$ *and* C *depends only on* u *and* T_0*.*

Proof:

Going back to the proof of theorem 2.1, we see that (26) can be replaced by

$$\|(u^n-u)(t)\|_{L^\infty} \le C\|u^n-u(t)\|_{H^{s-\beta}} \le C((\nu_n t)^{\beta/2}+\|u_0^n-u_0\|_{H^{s-\beta}}). \tag{30}$$

Hence (28) can be replaced by

$$\|(u^n-u)(t)\|_{H^s} \le Ct(\frac{(\nu t)^{\beta/2}+\|u_0^n-u_0\|_{H^{s-\beta}}}{\delta}+\frac{\nu}{\delta^2})+ C(\varepsilon_t(\delta)+\|u_0^n-u_0\|_{H^s}). \tag{31}$$

Taking the optimum in δ and applying lemma 2.5, we deduce easily that (29) holds. □

Lemma 2.5 *For* $a, b, t > 0$*, we have*

$$\inf_{\delta>0}\frac{a}{\delta}+\frac{b}{\delta^2}+\varepsilon_t(\delta) \le 2\frac{a}{f_t^{-1}(a)}+\frac{b}{(g_t^{-1}(b))^2} \tag{32}$$

The proof of this lemma is simple and is left for the reader

If we assume that u_0 is more regular, we can give a more precise rate.

Corollory 2.6 *We take the same hypotheses as in theorem 2.1 and assume in addition that* $u_0 \in H^{s+\alpha}$ *for some* $0 < \alpha \le 2$*.*

If $1 \le \alpha \le 2$*, we have*

$$\|(u^n-u)(t)\|_{H^s} \le C((\nu t)^{\alpha/2}+\|u_0^n-u_0\|_{H^s}) \tag{33}$$

for all $0 \le t \le T_0$ *and* C *depends only on* u *and* T_0*.*

If $0 < \alpha < 1$*, then for all* β *such that* $1 \le \alpha+\beta \le 2$ *and* $s-\beta > d/2$*, we have*

$$\|(u^n-u)(t)\|_{H^s} \le C(t\|u_0^n-u_0\|_{H^{s-\beta}})^\alpha + C((\nu t)^{\alpha/2}+\|u_0^n-u_0\|_{H^s}) \tag{34}$$

for all $0 \le t \le T_0$ *and* C *depends only on* u*,* β *and* T_0*.*

Proof:

First, notice that from the extra regularity of u_0, we deduce that $\|v^\delta\|_{H^{s+1}} \leq C(1+\delta^{\alpha-1})$, $\|v^\delta\|_{H^{s+2}} \leq C\delta^{\alpha-2}$ and $\|v^\delta - u\|_{H^s} \leq C\delta^\alpha$.

If $1 \leq \alpha \leq 2$, then (24) yields

$$\partial_t \|w\|_{H^s} \leq C\|w\|_{H^s} + C\nu_n \delta^{\alpha-2}. \tag{35}$$

Hence

$$\|(u^n - u)(t)\|_{H^s} \leq C\Big(\nu t \delta^{\alpha-2} + \delta^\alpha + \|u_0^n - u_0\|_{H^s}\Big) \tag{36}$$

Taking the optimum in δ, namely $\delta = \sqrt{\nu t}$, we deduce that (33) holds.

If $0 < \alpha < 1$, then arguing as in (15), we have

$$\begin{aligned} &\partial_t \|w^n\|^2_{H^{s-\beta}} + \nu_n \|\nabla w^n\|^2_{H^{s-\beta}} \\ &\leq \Big(C(\|u\|_{H^{s+\alpha}} + \|w^n\|_{H^s})\|w^n\|^2_{H^{s-\beta}} + \nu_n \|\nabla u\|_{H^{s+\alpha-1}} \|\nabla w^n\|_{H^{s+1-2\beta-\alpha}}\Big). \end{aligned} \tag{37}$$

Then using an interpolation inequality and Holder inequality, we deduce that

$$\|\nabla w^n\|_{H^{s+1-2\beta-\alpha}} \leq C\|w^n\|^{\beta+\alpha-1}_{H^{s-\beta}} \|\nabla w^n\|^{2-\beta-\alpha}_{H^{s-\beta}} \leq \frac{1}{C}\|\nabla w^n\|^2_{H^{s-\beta}} + C^2 \|w^n\|^{2-\frac{2}{\beta+\alpha}}_{H^{s-\beta}}.$$

Hence, we deduce that

$$\|(u^n - u)(t)\|_{H^{s-\beta}} \leq C((\nu_n t)^{(\beta+\alpha)/2} + \|u_0^n - u_0\|_{H^{s-\beta}}). \tag{38}$$

In the proof of theorem 2.1, we see that (26) can be replaced by

$$\|(u^n - u)(t)\|_{L^\infty} \leq C\|u^n - u(t)\|_{H^{s-\beta}} \leq C((\nu_n t)^{(\beta+\alpha)/2} + \|u_0^n - u_0\|_{H^{s-\beta}}). \tag{39}$$

Moreover, $\|v^\alpha - u\|_{L^\infty} \leq C\delta^{\beta+\alpha}$. Hence (28) can be replaced by

$$\begin{aligned} \|(u^n - u)(t)\|_{H^s} \leq Ct\Big(((\nu t)^{(\beta+\alpha)/2} + \|u_0^n - u_0\|_{H^{s-\beta}})\delta^{\alpha-1} + \nu\delta^{\alpha-2}\Big) + \\ C(\delta^\alpha + t\delta^{\beta+2\alpha-1} + \|u_0^n - u_0\|_{H^s}). \end{aligned} \tag{40}$$

Taking the optimum in δ, we deduce that

$$\|(u^n - u)(t)\|_{H^s} \leq C\Big(t((\nu t)^{(\beta+\alpha)/2} + \|u_0^n - u_0\|_{H^{s-\beta}})\Big)^\alpha + C(\nu t)^{\alpha/2} + C\|u_0^n - u_0\|_{H^s}. \tag{41}$$

Hence (34) holds. □

3 Vortex patches

In this section $d = 2$ or 3. For $2D$ vortex patches, namely the case where $\mathrm{curl}(u_0)$ is the characteristic function of a $C^{1+\alpha}$ domain $\alpha > 0$, it was proved in [7] (see also [4]) that the Euler system (6,7,8) has a unique solution u such that the characteristic function of $\mathrm{curl}(u)$ remains a $C^{1+\alpha}$ domain and that the velocity u is in $L^\infty_{loc}(\mathbb{R}; Lip)$. A similar result holds for $3D$ vortex patches, but only on a bounded interval, namely $u \in L^\infty_{loc}(0, T^*; Lip)$ (see [14]).

For vortex patches, theorem 2.1 does not apply. Indeed, the velocity is not in H^s for any $s > d/2 + 1$. For $2D$ vortex patches, it was proved in [11, 12] that the convergence to the Euler system still holds and that

$$\|u^n - u\|_{L^\infty(0,T;L^2)} \leq C(\nu_n T)^{\frac{1}{2}}. \tag{42}$$

In [12], the authors also prove some estimate in L^p spaces for the difference between the vorticities, in particular they prove for $p \geq 2$ that $\|\mathrm{curl}(u^n - u)\|_{L^\infty(0,T;L^p)} \leq C\nu_n^{\frac{1}{4p}-\varepsilon}$ for some short time T and $\varepsilon > 0$.

Also, in [1], a better rate of convergence is given for $2D$ vortex patches, namely

$$\|u^n - u\|_{L^\infty(0,T;L^2)} \leq C(\nu_n T)^{\frac{3}{4}} \tag{43}$$

which is optimal.

Here, we would like to extend the result of Abid and Danchin [1] to the $3D$ case and also give a slight improvement of their $2D$ result by allowing $u_0^n - u_0$ to be just in L^2. Moreover, the proof we present is much simpler.

Let us recall the definition of a vortex patch

Definition 3.1 *Take $0 < r < 1$. A vector field u is called a C^r vortex patch if the following decomposition holds*

$$\mathrm{curl}(u) = \chi_P \omega_i + \chi_{P^c} \omega_e \tag{44}$$

where $P \subset \mathbb{R}^d$ is an open set of class C^{1+r} and $\omega_i, \omega_e \in C^r(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ are compactly supported.

Here χ_P denotes the characteristic function of P. Notice that since $\mathrm{curl}(u)$ is divergence-free, we have $\omega_i.n = \omega_e.n$ on ∂P. This condition is always satisfied if $d = 2$.

First, we recall that Gamblin and Saint-Raymond [14] proved the existence of a local solution $u \in L^\infty_{loc}(0, T^*; Lip)$ to the vortex patch problem in $3D$ (see also [13] and [15]). Moreover, u remains a C^r vortex patch.

Hence, $\mathrm{curl}(u) \in L^\infty_{loc}(0,T^*;\dot{B}^\alpha_{2,\infty})$ where $\alpha = \min(r,1/2)$ (see the appendix).

Theorem 3.2 *Here $d = 2$ or 3. We assume that $u_0^n - u_0$ goes to 0 in $L^2(\mathbb{R}^d)$ when n goes to infinity. We also assume that u_0 is a C^r vortex patch.*

Then, if T^* is the time of existence and $u \in C_{loc}([0,T^*); Lip)$ is the solution of the Euler system with initial data u_0, then for all $0 < T < T^*$, there exists ν_0 such that for all $\nu_n \le \nu_0$ and for all sequence of weak (Leray) solutions to the Navier-Stokes system (3 - 5), we have for $0 < t < T$,

$$\|(u^n - u)(t)\|_{L^2} \le C((t\nu_n)^{\frac{1+\alpha}{2}} + \|u_0^n - u_0\|_{L^2}) \tag{45}$$

where $\alpha = \min(1/2, r)$ and where C depends only on u and T.

Remark 3.3 *In the $2D$ case, knowing that $\mathrm{curl}(u_0) \in L^1 \cap L^\infty$ does not imply that $u \in L^2$ and in general u_0 is not in L^2 unless $\int \mathrm{curl} u = 0$. In particular in the classical $2D$ vortex patch problem [18], namely the case curl(u) is the characteristic function of a C^{r+1} domain, u_0 is not in L^2. However, in the $3D$ case, the fact that $\mathrm{curl} u_0 \in L^1 \cap L^\infty$ implies that $u_0 \in L^2$ from Biot-Savart formula.*

Proof: Let us denote $w^n = u^n - u$, hence

$$\partial_t w^n + w^n.\nabla u + u.\nabla w^n - \nu_n \Delta w^n - \nu_n \Delta u = -\nabla(p^n - p). \tag{46}$$

Taking the L^2 product with w^n, we get (at least formally) for $0 < t < T$,

$$\frac{1}{2}\|w^n(t)\|^2_{L^2} + \nu_n \int_0^t \|\nabla w^n\|^2_{L^2} \le \frac{1}{2}\|w^n(0)\|^2_{L^2} + \int_0^t C\|\nabla u\|_{L^\infty}\|w^n\|^2_{L^2} - \int_0^t \int \nu_n \nabla u.\nabla w^n. \tag{47}$$

In the $2D$ case, this computation is fully justified. We only point that in the $2D$ case, u and u^n are not in general in L^2 but their difference is in L^2.

To prove (47) rigorously in the $3D$ case, we just add the energy inequality (48) and energy equality (49)

$$\frac{1}{2}\|u^n(t)\|^2_{L^2} + \nu_n \int_0^t \|\nabla u^n\|^2_{L^2} \le \frac{1}{2}\|u^n(0)\|^2_{L^2} \tag{48}$$

$$\frac{1}{2}\|u(t)\|_{L^2}^2 = \frac{1}{2}\|u(0)\|_{L^2}^2 \tag{49}$$

and subtract the weak formulation

$$\int u^n u(t) - \int u^n u(0) + \int_0^t \int u^n u.\nabla u + uu^n.\nabla u^n + \nu_n \nabla u^n.\nabla u = 0. \tag{50}$$

Besides, using the duality between $\dot{B}_{2,1}^{-\alpha}$ and $\dot{B}_{2,\infty}^{\alpha}$, the divergence-free property of u and lemma 5.1 (see the appendix), we have

$$\begin{aligned} |\int \nabla u.\nabla w^n| &\leq C\|\nabla u\|_{\dot{B}_{2,\infty}^{\alpha}} \|\nabla w^n\|_{\dot{B}_{2,1}^{-\alpha}} &(51)\\ &\leq C\|\mathrm{curl} u\|_{\dot{B}_{2,\infty}^{\alpha}} \|w^n\|_{\dot{B}_{2,1}^{1-\alpha}} &(52)\\ &\leq C\|\mathrm{curl} u\|_{\dot{B}_{2,\infty}^{\alpha}} \|w^n\|_{L^2}^{\alpha} \|\nabla w^n\|_{L^2}^{1-\alpha}. &(53)\end{aligned}$$

By Holder inequality, we have

$$\nu_n \|\mathrm{curl} u\|_{\dot{B}_{2,\infty}^{\alpha}} \|w^n\|_{L^2}^{\alpha} \|\nabla w^n\|_{L^2}^{1-\alpha} \leq C\nu_n \|\mathrm{curl} u\|_{\dot{B}_{2,\infty}^{\alpha}}^{\frac{2}{1+\alpha}} \|w^n\|_{L^2}^{\frac{2\alpha}{1+\alpha}} + \frac{\nu_n}{2}\|\nabla w^n\|_{L^2}^2. \tag{54}$$

Hence, we get from (47)

$$\|w^n(t)\|_{L^2}^2 \leq \|w^n(0)\|_{L^2}^2 + C\int_0^t \|\nabla u\|_{L^\infty} \|w^n\|_{L^2}^2 + C\nu_n \|\mathrm{curl} u\|_{\dot{B}_{2,\infty}^{\alpha}}^{\frac{2}{1+\alpha}} \|w^n\|_{L^2}^{\frac{2\alpha}{1+\alpha}} ds. \tag{55}$$

And by Grönwall lemma, we deduce that

$$\|w^n(t)\|_{L^2}^{\frac{2}{1+\alpha}} \leq C\|w^n(0)\|_{L^2}^{\frac{2}{1+\alpha}} + C\nu_n t \tag{56}$$

and (45) follows. □

From the proof, we can see that the only information we used about u is that $u \in L^\infty_{loc}(0, T^*; Lip)$ and curl$u \in L^\infty_{loc}(0, T^*; \dot{B}_{2,\infty}^{\alpha})$. Moreover, it is easy to see that if $u \in L^\infty_{loc}([0, T^*); Lip)$ then the $\dot{B}_{2,\infty}^{\alpha}$, $0 < \alpha < 1$ regularity of curl$u$ is propagated by the flow, namely if curl$u^0 \in \dot{B}_{2,\infty}^{\alpha}$, then curl$u \in L^\infty_{loc}([0, T^*); \dot{B}_{2,\infty}^{\alpha})$ (see [8]). Hence, we have the following theorem

Theorem 3.4 *We assume that* $u_0^n - u_0$ *goes to* 0 *in* $L^2(\mathbb{R}^d)$ *when* n *goes to infinity and that* $\mathrm{curl} u_0 \in \dot{B}^{\alpha}_{2,\infty}$, $0 < \alpha < 1$. *We also assume that the Euler system with the initial data* u_0 *has a unique solution* $u \in L^\infty_{loc}([0, T^*); Lip)$. *Then for all* $0 < T < T^*$, *there exists* ν_0 *such that for all* $\nu_n \leq \nu_0$ *and for all sequence of weak (Leray) solutions to the Navier-Stokes system (3 - 5), we have for* $0 < t < T$,

$$\|(u^n - u)(t)\|_{L^2} \leq C((t\nu_n)^{\frac{1+\alpha}{2}} + \|u_0^n - u_0\|_{L^2}) \tag{57}$$

where C *depends only on* u *and* T.

This theorem is an improvement of theorem 1.1 of [1] since we only assume that the solution of the Euler system is Lipschitz. We would like to give two applications of this theorem which yield a better convergence rate than the simple application of theorem 3.2.

Consider a vector field u_0 which is a C^r vortex patch with $0 < r < 1/2$ and assume in addition that $\mathrm{curl} u^0 \in \dot{B}^{1/2}_{2,\infty}$, then theorem 3.4 allows us to prove that

$$\|(u^n - u)(t)\|_{L^2} \leq C((t\nu_n)^{\frac{3}{4}} + \|u_0^n - u_0\|_{L^2}) \tag{58}$$

which is better than the rate we get from (45).

There are several situations where u_0 is a C^r vortex patch with $0 < r < 1/2$ and $\mathrm{curl} u_0 \in \dot{B}^{1/2}_{2,\infty}$. For instance this is the case if $\mathrm{curl}(u_0) = \chi_P \omega_{i0} + \chi_{P^c}\omega_{e0}$ is such that $P \subset \mathbb{R}^d$ is an open set of class C^{1+r} and $\omega_{i0}, \omega_{e0} \in C^{1/2}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. We notice here that for $t > 0$ we only know that u is a C^r vortex patch, namely $\mathrm{curl}(u) = \chi_{P(t)}\omega_i(t) + \chi_{P(t)^c}\omega_e(t)$ with $P(t)$ of class C^{1+r} and $\omega_i(t), \omega_e(t) \in L^\infty_{loc}(0, T^*; C^r(\mathbb{R}^d) \cap L^1(\mathbb{R}^d))$. Hence, $\mathrm{curl} u \in L^\infty_{loc}([0, T^*); \dot{B}^r_{2,\infty})$. However, propagating the initial $\dot{B}^{1/2}_{2,\infty}$ yields that $\mathrm{curl} u \in L^\infty_{loc}([0, T^*); \dot{B}^{1/2}_{2,\infty})$ and gives the better rate (58).

In particular theorem 3.4 applies to the classical $2D$ vortex patch, namely the case $\mathrm{curl} u_0 = \chi_P$ and P is of class C^{1+r}, $r > 0$ in which case (58) holds even if $r < 1/2$.

Remark 3.5 *In the* $2D$ *case, one can lower the regularity of the initial data. Indeed Yudovich [22] proved that if* $\omega_0 = \mathrm{curl}(u_0) \in L^\infty \cap L^p$ *for some* $1 < p < \infty$ *then the Euler system (6) has a unique global solution (see also [8]). It was proved in [9] that the solution to the Navier-Stokes system converges* *in $L^\infty((0,T);L^2)$ to the solution of the Euler system if we only assume that $\omega_0=\mathrm{curl}(u_0)\in L^\infty\cap L^p$. More precisely, Chemin [9] proves that*

$$\|u^n-u\|_{L^\infty(0,T;L^2)}\le C\|\mathrm{curl}(u_0)\|_{L^\infty\cap L^2}(\nu_nT)^{\frac{1}{2}exp(-C\|\mathrm{curl}(u_0)\|_{L^\infty\cap L^2}T)}. \quad (59)$$

Notice that here, the rate of convergence deteriorates with time. This does not happen if we also know that u is in $L^\infty(0,T;Lip)$ as was proved by Constantin and Wu [11].

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5 Appendix

We define $\mathcal{C}$ to be the ring of center 0, of small radius 1/2 and great radius 2. There exist two nonnegative radial functions χ and φ belonging respectively to $\mathcal{D}(B(0,1))$ and to $\mathcal{D}(\mathcal{C})$ so that

$$\chi(\xi)+\sum_{q\ge 0}\varphi(2^{-q}\xi)=1, \quad (60)$$

$$|p-q|\ge 2\Rightarrow \mathrm{Supp}\ \varphi(2^{-\mathrm{q}}\cdot)\cap \mathrm{Supp}\ \varphi(2^{-\mathrm{p}}\cdot)=\emptyset. \quad (61)$$

For instance, one can take $\chi\in\mathcal{D}(B(0,1))$ such that $\chi\equiv 1$ on $B(0,1/2)$ and take

$$\varphi(\xi)=\chi(2\xi)-\chi(\xi).$$

Then, we are able to define the Littlewood-Paley decomposition. Let us denote by $\mathcal{F}$ the Fourier transform on $\mathbb{R}^d$. Let $h,\ \widetilde{h},\ \Delta_q,\ S_q\ (q\in\mathbb{Z})$ be defined as follows:

$$h=\mathcal{F}^{-1}\varphi \quad \text{and} \quad \widetilde{h}=\mathcal{F}^{-1}\chi,$$
$$\Delta_q u=\mathcal{F}^{-1}(\varphi(2^{-q}\xi)\mathcal{F}u)=2^{qd}\int h(2^qy)u(x-y)dy,$$
$$S_q u=\mathcal{F}^{-1}(\chi(2^{-q}\xi)\mathcal{F}u)=2^{qd}\int \widetilde{h}(2^qy)u(x-y)dy.$$

Then, we define the non-homogeneous and homogeneous Besov norms

$$\|u\|_{B^s_{2,r}} = \Big(\|S_0 u\|^r_{L^2} + \sum_{q\geq 0} 2^{rsq}\|\Delta_q u\|^r_{L^2}\Big)^{1/r}$$

$$\|u\|_{\dot{B}^s_{2,r}} = \Big(\sum_{q\in\mathbb{Z}} 2^{rsq}\|\Delta_q u\|^r_{L^2}\Big)^{1/r}$$

for $s \in \mathbb{R}$ and $1 \leq r \leq \infty$. If $r = \infty$, then the summation over q is replaced by the L^∞ norm.

Lemma 5.1 *For* $0 < \alpha < 1$, *we have*

$$\|w\|_{\dot{B}^{1-\alpha}_{2,1}} \leq C\|w\|^\alpha_{L^2}\|\nabla w\|^{1-\alpha}_{L^2}. \tag{62}$$

Proof:

This inequality can be easily deduced from real interpolation. We give here a direct proof. Actually, we will prove a stronger estimate, namely

$$\|w\|_{\dot{B}^{1-\alpha}_{2,1}} \leq C\|w\|^\alpha_{\dot{B}^0_{2,\infty}}\|\nabla w\|^{1-\alpha}_{\dot{B}^0_{2,\infty}}. \tag{63}$$

Indeed, we have

$$\|\Delta_q w\|_{L^2} \leq C\|w\|_{\dot{B}^0_{2,\infty}} \tag{64}$$

$$\|\Delta_q w\|_{L^2} \leq C2^{-q}\|w\|_{\dot{B}^1_{2,\infty}}. \tag{65}$$

We take N such that

$$2^N\|w\|_{\dot{B}^0_{2,\infty}} \leq \|w\|_{\dot{B}^1_{2,\infty}} \leq 2^{N+1}\|w\|_{\dot{B}^0_{2,\infty}}.$$

Hence

$$\begin{aligned}\sum_{q=-\infty}^{\infty} 2^{(1-\alpha)q}\|\Delta_q w\|_{L^2} &\leq C\sum_{q\leq N} 2^{(1-\alpha)q}\|w\|_{\dot{B}^0_{2,\infty}} + C\sum_{q\geq N} 2^{-\alpha q}\|w\|_{\dot{B}^1_{2,\infty}} \\ &\leq C2^{(1-\alpha)N}\|w\|_{\dot{B}^0_{2,\infty}} + C2^{-\alpha N}\|w\|_{\dot{B}^1_{2,\infty}}.\end{aligned} \tag{66}$$

From which (63) follows. □

In the next two lemmas, we prove that if u_0 is a C^r vortex patch then $\mathrm{curl}(u) \in L^\infty_{loc}(0, T^*; B^\alpha_{2,\infty})$ where $\alpha = \min(r, 1/2)$.

Lemma 5.2 *If P is bounded open set of $\mathbb{R}^d$ of class C^{1+r} then $\chi_{_P} \in \dot{B}^{1/2}_{2,\infty}$*

The proof is based on interpolation. Indeed, since P is C^{1+r}, it is Liptchiz and hence $\chi_{_P} \in L^\infty \cap BV$. Then

$$\|\Delta_q \chi_{_P}\|_{L^\infty} \leq C\|\chi_{_P}\|_{L^\infty} \tag{67}$$

$$\|\Delta_q \chi_{_P}\|_{L^1} \leq C2^{-q}\|\chi_{_P}\|_{BV}. \tag{68}$$

interpolating between L^1 and L^∞, we deduce that

$$\|\Delta_q \chi_{_P}\|_{L^2} \leq C2^{-q/2}\|\chi_{_P}\|_{BV}\|\chi_{_P}\|_{L^\infty}. \tag{69}$$

and hence, $\chi_{_P} \in \dot{B}^{1/2}_{2,\infty}$. □

Lemma 5.3 *If u is a C^r vortex patch then* $\mathrm{curl}(u) \in \dot{B}^{\alpha}_{2,\infty}$ *where* $\alpha = \min(r, 1/2)$.

The proof uses the para-product decomposition of Bony ([6])

$$uv = T_u v \ + \ T_v u \ + \ R(u,v)$$

where

$$T_u v = \sum_{q\in\mathbb{Z}} S_{q-1} u \Delta_q v \quad \text{and} \quad R(u,v) = \sum_{|q-q'|\leq 1} \Delta_{q'} u \Delta_q v.$$

We decompose

$$\chi_{_P}\omega_i = T_{\omega_i}\chi_{_P} + R(\chi_{_P}, \omega_i) + T_{\chi_P}\omega_i \tag{70}$$

and notice that since $\chi_{_P}$ and ω_i are both in L^∞, we get

$$\|T_{\omega_i}\chi_{_P} + R(\chi_{_P}, \omega_i)\|_{\dot{B}^{\alpha}_{2,\infty}} \leq C\|\omega_i\|_{L^\infty}\|\chi_{_P}\|_{\dot{B}^{\alpha}_{2,\infty}} \tag{71}$$

$$\|T_{\chi_P}\omega_i\|_{\dot{B}^{\alpha}_{2,\infty}} \leq C\|\chi_{_P}\|_{L^\infty}\|\omega_i\|_{\dot{B}^{\alpha}_{2,\infty}} \tag{72}$$

Since, ω_i is in C^r and is compactly supported, we deduce that $\omega_i \in \dot{B}^{\alpha}_{2,\infty}$. Hence $\chi_{_P}\omega_i \in \dot{B}^{\alpha}_{2,\infty}$

The same proof holds for $\chi_{_{P^c}}\omega_e$ and hence, $\mathrm{curl}(u) \in \dot{B}^{\alpha}_{2,\infty}$. □

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