MINIMUM AREA ISOSCELES CONTAINERS

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ABSTRACT. We show that every minimum area isosceles triangle containing a given triangle T shares a side and an angle with T. This proves a conjecture of Nandakumar motivated by a computational problem. We use our result to deduce that for every triangle T, (1) there are at most 3 minimum area isosceles triangles that contain T, and (2) there exists an isosceles triangle containing T whose area is smaller than $\sqrt{2}$ times the area of T. Both bounds are best possible.

1. INTRODUCTION

Given two convex bodies, T' and T, in the plane, it is not easy to decide whether there is a rigid motion that takes T' into a position where it covers T. Suppose, for instance, that we place a 2-dimensional convex body T' in the 3-dimensional space, and let T denote the orthogonal projection of T' onto the x-y plane. The area of T' is at least as large as the area of T, and it looks plausible that T' can be moved to cover T. However, the proof of this fact is far from straightforward; see [3, 12]. As Steinhaus [21] pointed out, it is not even clear how to decide, whether a given triangle T' can be brought into a position where it covers a fixed triangle T. The first such algorithm was found by Post [17] in 1993, and it was based on the following lemma.

Lemma 1.1 (Post). *If a triangle T' can be moved to a position where it covers another triangle T, then one can also find a covering position of T' with a side that contains one side of T.*

In many problems, the body T' is not fixed, but can be chosen from a family of possible "containers," and we want to find a container which is in some sense optimal. To find a minimum area or minimum perimeter triangle, rectangle, convex k-gon, or ellipse (Löwner-John ellipse) enclosing a given set of points are classical problems in geometry with interesting applications in packing and covering, approximation, convexity, computational geometry, robotics, and elsewhere [1, 2, 4, 5, 6, 8, 9, 10, 16, 18, 19]. Finding optimal circumscribing and inscribed simplices, ellipsoids, polytopes with a fixed number of sides or vertices, etc., are fundamental questions in optimization, functional analysis, and number theory; see e.g. [7, 13, 11, 20, 22].

Motivated by a computational problem, R. Nandakumar [15] raised the following interesting special instance of the above question: *Determine the minimum area of an isosceles triangle containing a given triangle T.* The aim of the present note is to solve this problem and to find all triangles for which the minimum is attained. We call these triangles *minimum area isosceles containers* for T. It is easy to verify that every triangle has at least one minimum area isosceles container (see Lemma 3.1). However, we will see that in some cases the minimum area isosceles container is not unique.

Our main objective is to prove the following statement conjectured by Nandakumar [15].

Theorem 1.2. *Let T be a triangle and let $T' \supseteq T$ be one of its minimum area isosceles containers. Then, T' and T have a side in common, and their angles at one of the endpoints of this side are equal.*

For any two points, A and B, let AB denote the closed segment connecting them, and let $|AB|$ stand for the length of AB. To unify the presentation, in the sequel we fix a triangle T with vertices A, B, C, and side lengths $a = |BC|$, $b = |AC|$, $c = |AB|$. If two sides are of the same length, then T is the unique minimum area isosceles container of itself, so there is nothing to prove. Therefore, from now on we assume without loss of generality that $a < b < c$.

To establish Theorem 1.2 and to formulate our further results, we need to introduce some special isosceles triangles associated with the triangle ABC, each of which shares a side and an angle with ABC.

Special containers of the first kind. Let B' denote the point on the ray $\vec{CB}$, for which $|B'C| = |AC| = b$ (see Fig. 1). Analogously, let C' (and C'') denote the points on $\vec{AC}$ (resp., $\vec{BC}$) such that $|AC'| = c$ (resp., $|BC''| = c$). Obviously, the triangles $AB'C$, ABC', and ABC'' are isosceles. We call them special containers of the first kind associated with ABC.

Special containers of the second kind. Let B_1 denote the point on the ray $\vec{AB}$, different from A, for which $|B_1C| = |AC| = b$ (see Figure 2). Analogously, let C_1 (resp., C_2) denote the point on $\vec{AC}$

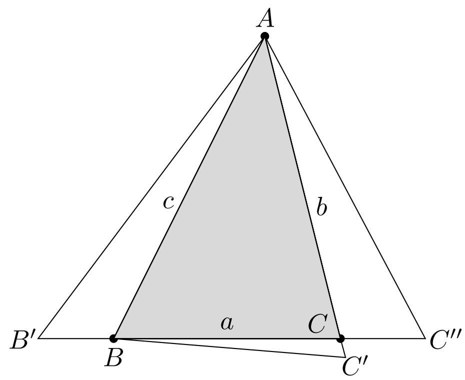

FIGURE 1. Special containers of the first kind $AB'C, ABC'$, and ABC''.

(resp., $\overrightarrow{BC}$) for which $|BC_1| = |AB| = c$ and $C_1 \neq A$ (resp., $|AC_2| = |AB| = c$ and $C_2 \neq B$). The triangles AB_1C, ABC_1, and ABC_2 are called the special containers of the second kind associated with ABC.

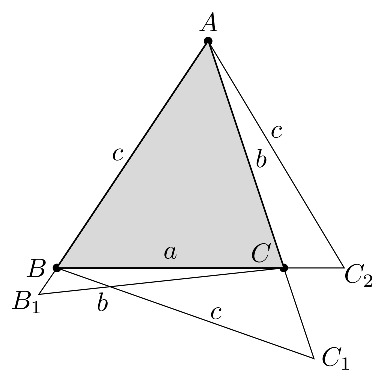

FIGURE 2. Special containers of the second kind AB_1C, ABC_1, and ABC_2.

Special containers of the third kind. Let $\overline{A}$ be the intersection of the perpendicular bisector of BC and the line AC. Since we have $b = |AC| < |AB| = c$, the point $\overline{A}$ lies outside of ABC. Analogously, denote by $\overline{B}$ (resp., $\overline{C}$) the intersection of the perpendicular bisector of AC (resp. AB) and the line BC. Note that $\overline{A}BC$ and $A\overline{B}C$ do not contain ABC if $\sphericalangle BCA \geq 90°$. The triangles $\overline{A}BC$, $A\overline{B}C$, and $AB\overline{C}$ are called special containers of the third kind associated with ABC, provided that they contain ABC. Thus, if ABC is acute, then it has three special containers of the third kind. Otherwise, it has only one (see Figure 3).

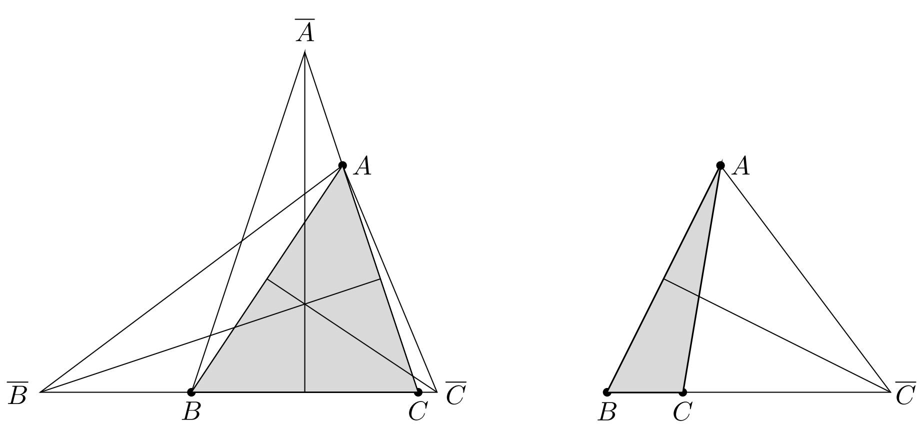

FIGURE 3. Special containers of the third kind in the acute and in the non-acute cases.

All special containers share a common angle and a common side with the original triangle ABC. Obviously, there is no other isosceles container having the same property. Indeed, for each vertex of ABC, there are at most 3 isosceles triangles that share this vertex and the angle at this vertex with ABC, and also have a common side with ABC.

Therefore, Theorem 1.2 is an immediate corollary of the following statement.

Theorem 1.3. *All minimum area isosceles containers for a triangle are special containers of the first kind, or of the second kind, or of the third kind.*

Whenever a minimum area isosceles container of a triangle is acute, we can be more specific.

Theorem 1.4. *If a minimum area isosceles container of a triangle is acute, then it is a special container of the first kind.*

One is tempted to believe that if a triangle is acute, then all of its minimum area isosceles containers are acute and, hence, all of them are special containers of the first kind. However, this is not the case: Example 5.1 demonstrates that there are acute triangles with obtuse minimum area isosceles containers. As all special containers of the first kind and the third kind of an acute triangle are acute, Theorem 1.4 implies the following statement.

Corollary 1.5. *A minimum area isosceles container for an acute triangle is obtuse if and only if it is a special container of the second kind.*

It follows from Theorem 1.3 that every triangle has at most 9 minimum area isosceles containers: at most 3 special containers of each kind. In the next section, we prove that there are no minimum area isosceles triangles of the third kind (see Lemma 2.2). Thus, every triangle can have at most 6 minimum area isosceles triangles. In fact, this bound can be further reduced to 3.

Theorem 1.6. *Every non-isosceles triangle ABC has at most 3 minimum area isosceles containers, $AB'C$, ABC', and AB_1C. In particular, every minimum area isosceles container is a special container of the first or the second kind.*

There is a unique triangle T^*, up to similarity, which has precisely 3 different minimum area isosceles containers. Its angles are $\alpha^* \approx 41.831452°, 2\alpha^*$, and $180° - 3\alpha^*$, where α^* is the unique solution of $\sin(\alpha)\sin(2\alpha) - \sin^2(3\alpha) = 0$ in the interval $[36°, 45°]$.

Finally, we discuss how large the area of a minimum area isosceles container for a triangle T can be relative to the area of T. We also consider the same question for special containers of the first kind.

Theorem 1.7. *(a) Every triangle of area 1 has an isosceles container whose area is smaller than $\sqrt{2}$.*
(b) Every triangle of area 1 has a special container of the first kind, whose area is smaller than $\frac{1+\sqrt{5}}{2}$.

Both bounds are best possible.

As (b) is best possible, there exists a triangle of area 1 for which every special container of the first kind has area larger than $\frac{1+\sqrt{5}}{2}$. Therefore, by (a), none of its special containers of the first kind can be a minimum area isosceles container. This disproves an earlier conjecture of Nandakumar, according to which every triangle T admits a minimum area isosceles container which is a special container of the first kind.

Our paper is organized as follows. In Section 2, we prove some useful inequalities for the areas of special containers. In particular, we prove Theorem 1.7 (b). In Section 3, we establish some elementary properties of minimum area isosceles containers. Section 4 contains the proofs of Theorems 1.3, and 1.4. Finally, Theorem 1.6, and 1.7 (a) are proved in Section 5.

2. Preliminaries—Proof of Theorem 1.7 (b)

In this section, we collect some basic facts about special containers and establish Theorem 1.7 (b).

First, we consider special containers of the first kind, because their areas can be easily compared to the area of the triangle ABC. As everywhere else, we assume that the side lengths of ABC satisfy $a < b < c$. The area of ABC is denoted by $t(ABC)$.

Lemma 2.1. *For any non-isosceles triangle ABC, we have*

(a) $t(ABC'') > t(ABC')$,
(b) $t(AB'C) > t(ABC')$ *(resp.,* $t(AB'C) \geq t(ABC')$*)*
if and only if $b^2 > ac$ *(resp.,* $b^2 \geq ac$ *).*

Proof. Let m be the length of the altitude of ABC perpendicular to the side BC. We have $t(ABC) = \frac{a \cdot m}{2}$ and $t(AB'C) = \frac{b \cdot m}{2}$. Thus, the ratio $t(AB'C)/t(ABC) = b/a$.
Similar arguments show that

$$\frac{t(ABC')}{t(ABC)} = \frac{c}{b} \quad \text{and} \quad \frac{t(ABC'')}{t(ABC)} = \frac{c}{a}.$$

(a) Since $b > a$, we have $t(ABC'') > t(ABC')$.
(b) Straightforward. □

Next, using Lemma 2.1, we determine the supremum of the ratio of the area of the smallest isosceles containers of the first kind to the area of the original triangle ABC. This will also provide an upper bound for the ratio of the area of a smallest area *isosceles* containers to the area of the original triangle. This fact will be used in the sequel.

Proof of Theorem 1.7 (b). Let r_1^* denote the supremum of the ratio of the area of a smallest container of the first kind associated with ABC and the area of ABC, over all triangles ABC with the above property. We show that $r_1^* = \frac{1+\sqrt{5}}{2}$. Suppose without loss of generality that $a = 1$. By our assumptions and the triangle inequality, we have $1 < b < c < b+1$. Let

$$r(b,c) := \begin{cases} b & \text{if } b^2 \leq c, \\ c/b & \text{if } b^2 > c. \end{cases}$$

By Lemma 2.1, $r(b,c) = \min\left(\frac{t(AB'C)}{t(ABC)}, \frac{t(ABC')}{t(ABC)}\right)$, and

$$r_1^* = \sup_{1<b<c<b+1} r(b,c).$$

If $b^2 \leq c < b+1$, then $r(b,c) = b < \frac{1+\sqrt{5}}{2}$.

If $b^2 > c$ and $b < \frac{1+\sqrt{5}}{2}$, then $r(b,c) = \frac{c}{b} < b < \frac{1+\sqrt{5}}{2}$. Otherwise, if $b \geq \frac{1+\sqrt{5}}{2}$ (and, hence, $b^2 > c$), then $r(b,c) = \frac{c}{b} < \frac{b+1}{b} = 1 + \frac{1}{b} \leq 1 + \frac{2}{1+\sqrt{5}} = \frac{1+\sqrt{5}}{2}$. Thus, we obtain that $r(b,c) < r_1^* \leq \frac{1+\sqrt{5}}{2}$, for all $1 < b < c < b+1$, i.e, the supremum r_1^* is not attained for any triangle ABC.

The supremum of $r(b,c)$, restricted to the parabola arc $c = b^2 < b+1$ in the (b,c) plane, is $\frac{1+\sqrt{5}}{2}$. Since every point (b,c) of this arc corresponds to a triangle with side lengths $1, b, c$, we obtain that $r_1^* = \frac{1+\sqrt{5}}{2}$, as required. □

We show that for any special container of the third kind, there exists a special container of the second kind whose area is smaller.

Lemma 2.2. *For any triangle ABC we have $t(AB_1C) < t(AB\overline{C})$. If ABC is acute, then we also have $t(ABC_1) < t(A\overline{B}C)$ and $t(ABC_2) < t(\overline{A}BC)$. Thus, a minimum area isosceles container can never be a special container of the third kind.*

Proof. We verify only the first inequality; the other two statements can be shown analogously.

Assign planar coordinates to the points. We can assume without loss of generality that $A = (0,0)$, $B = (2,0)$, $C = (p,q)$, and $\overline{C} = (1,d)$. Then $t(AB\overline{C}) = d$. The equation of the line passing through B, C, and $\overline{C}$ is $dx + y = 2d$. Taking $p = 1+s$ for some $0 < s < 1$, we have $q = d(1-s)$ and $B_1 = (2(1+s), 0)$. Hence, $t(AB_1C) = d(1-s^2) < d = t(AB\overline{C})$. □

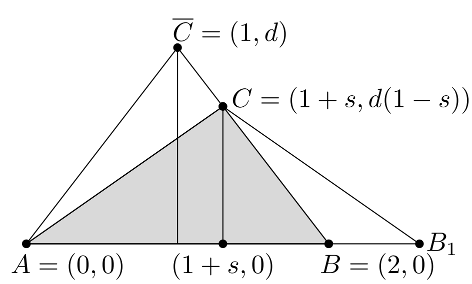

FIGURE 4. Proof of Lemma 2.2.

3. Four useful lemmas

The aim of this section is to prepare the ground for the proofs of the main results that will be given in the next two sections.

First, we show that the problem is well-defined, that is, for every triangle ABC, there is at least one isosceles triangle containing ABC, whose area is smaller than or equal to the area of any other isosceles container.

Lemma 3.1. *Every triangle ABC has at least one minimum area isosceles container.*

Proof. It follows from Theorem 1.7 (b) that the area of a minimum area isosceles container is at most $\frac{1+\sqrt{5}}{2}$ times larger than the area of the original triangle ABC. Therefore, the vertices of any minimum area isosceles container must lie within a bounded distance from ABC, and the statement follows by a standard compactness argument. □

Lemma 3.2. *Let ABC be a triangle and SPR a minimum area isosceles container for ABC. Then A, B, C must lie on the boundary of SPR, and each side of SPR contains at least one of them.*

Proof. First, we show that each side of SPR contains a vertex of ABC. Indeed, if one of the sides did not contain any vertex of ABC, then we could slightly move it, parallel to itself, to obtain a smaller isosceles container.

Assume next, for contradiction, that A, say, does not lie on the boundary of SPR. Then we could slightly rotate ABC about B or C, to bring it into a position where only one of its vertices lies on the boundary of SPR. However, in that case, at least one of the sides of SPR would contain no vertex of ABC. □

Let SPR be an isosceles triangle and let m_S, m_P, m_R denote the midpoints of the sides PR, SR, and SP, respectively. The boundary of SPR splits into three polygonal pieces, $\widehat{m_Pm_R}, \widehat{m_Rm_S}$ and $\widehat{m_Sm_P}$, each of which consists of two closed line segments. Namely,

$$\widehat{m_Pm_R} = m_PS \,\cup\, Sm_R,$$

$$\widehat{m_Rm_S} = m_RP \,\cup\, Pm_S,$$

$$\widehat{m_Sm_P} = m_SR \,\cup\, Rm_P.$$

See Figure 5.

Lemma 3.3. *Let ABC be a triangle and SPR a minimum area isosceles container for ABC. Then, each of the closed polygonal pieces $\widehat{m_Pm_R}, \widehat{m_Rm_S}$, and $\widehat{m_Sm_P}$ contains precisely one vertex of ABC.*

Proof. By Lemma 3.2, the vertices A, B, C lie on the boundary of SPR. Suppose for contradiction that the closed polygonal piece $\widehat{m_Pm_S}$ contains two vertices of ABC. We may and do assume without loss of generality that these vertices are A and C.

Let T_1 and T_2 denote the intersection points of the segment m_Pm_S with AB and CB, respectively. The quadrilateral $CAT_1T_2 \subseteq Rm_Pm_S$, so that $t(CAT_1T_2) \le t(Rm_Pm_S)$. Since $|T_1T_2| \le |m_Pm_S|$, we have $t(T_2T_1B) \le t(m_Sm_Pm_R)$. Consequently, we get

$$t(ABC) \le t(Rm_Pm_S) + t(m_Sm_Pm_R) = \frac{1}{2}t(SPR).$$

Equality holds if and only if $A = R, B = S, C = m_S$ or $A = m_P, B = P, C = R$.

On the other hand, by Theorem 1.7 (b), we obtain $t(SPR) < \frac{1+\sqrt{5}}{2}t(ABC)$, the desired contradiction. □

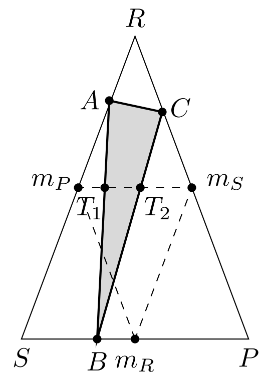

FIGURE 5. Illustration for the proof of Lemma 3.3.

Lemma 3.4. *Let ABC be a triangle and SPR a minimal area isosceles container for ABC. Then ABC and SPR have a common vertex.*

Proof. By Lemma 3.2, the points A, B, C must lie on the boundary SPR. Suppose for contradiction that none of them is a vertex of SPR. In view of Lemmas 3.2 and 3.3, there are two possibilities: each of the segments m_PR, m_RS, m_SP contains precisely one vertex of ABC, or each of the segments m_PS, m_RP, m_SR contains precisely one vertex of ABC. Suppose without loss of generality that $A \in m_PR, B \in m_RS, C \in m_SP$, as in Figure 6.

Let Q denote the center of the circle circumscribed around SPR. It is easy to see that a small clockwise rotation about Q will take ABC into a position such that all of its vertices lie in the interior of the triangle SPR. This contradicts the minimality of SPR and Lemma 3.2. □

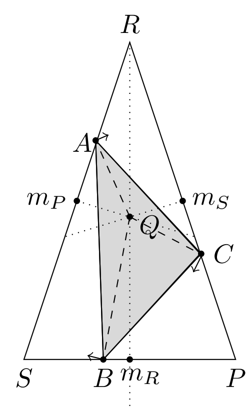

FIGURE 6. Illustration for the proof of Lemma 3.4.

4. NANDAKUMAR'S CONJECTURE—PROOFS OF THEOREMS 1.3 AND 1.4

We start with the proof of Theorem 1.3, which immediately implies Nandakumar's conjecture (Theorem 1.2).

Proof of Theorem 1.3. Let SPR be a minimum area container for ABC with apex R. By Lemma 3.2, A, B, and C are on the boundary of SPR, and, by Lemma 3.4, the triangles SPR and ABC share a vertex. Using Lemma 3.3 under the assumption that $ABC \neq SPR$, we can distinguish 8 cases, up to symmetry (see Figure 7). Cases (1)–(3) represent those instances when ABC and SPR have two common vertices. In these cases, SPR is a special container of the first, the second, and the third kind, respectively, so we are done.

In the remaining cases, ABC and SPR have only one vertex in common. In cases (4)–(6), this vertex is a base vertex (say, S) of SPR. Finally, in cases (7)–(8), R is the unique common vertex of ABC and SPR. It is sufficient to show that in cases (4)–(8), the area of SPR is not minimal.

First, we discuss cases (5)–(8). Case (4) is more delicate and is left to the end of the proof.

Cases (5) and (6) are analogous. Let D denote the vertex of ABC lying on RP. In both cases, we have $ABC \subseteq SPD$. Clearly, SPR is a special container of the third kind associated with SPD, and

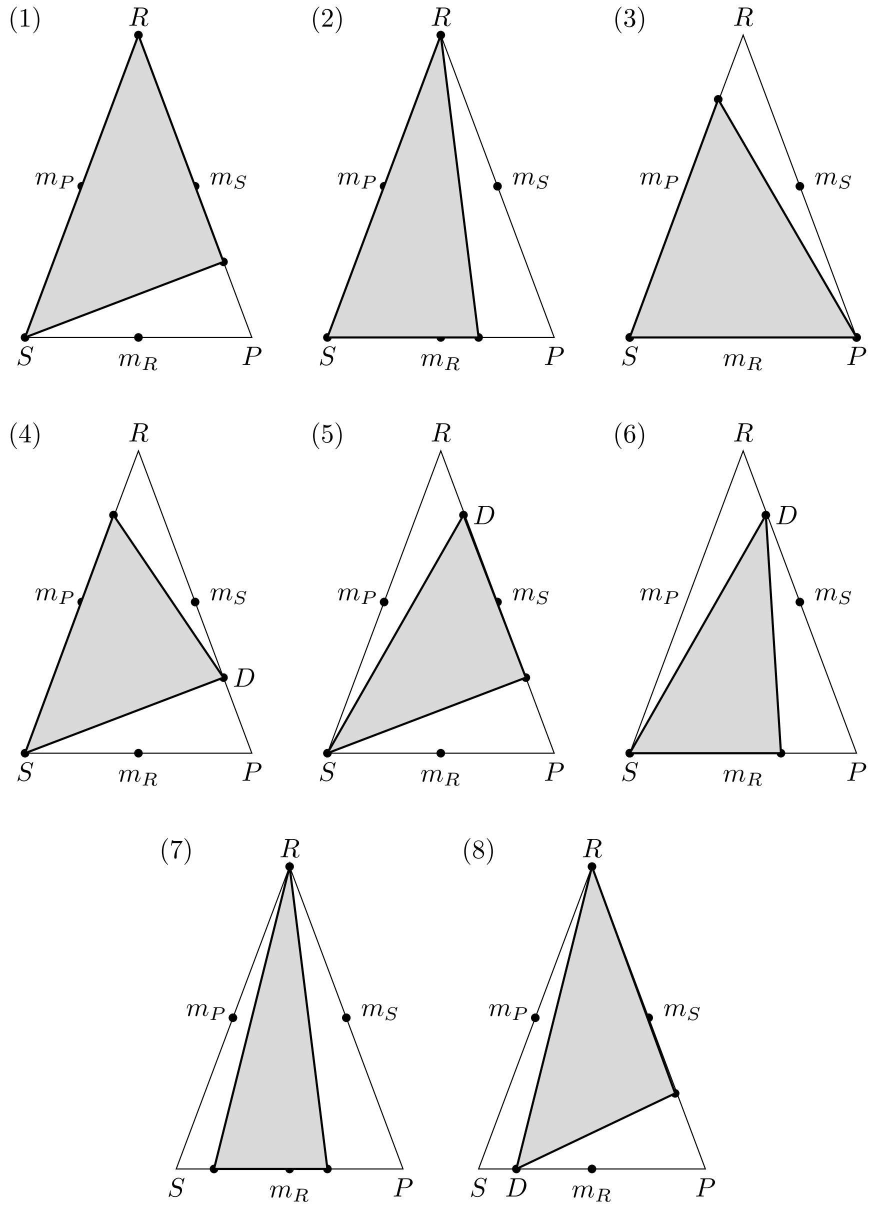

FIGURE 7. The 8 cases up to symmetry. Triangle ABC is shaded.

by Lemma 2.2, it cannot be minimal. Since every container for SPD is also a container for ABC, we conclude that SPR is not a minimum area container for ABC.

In case (7), we can find an isosceles triangle with apex R which contains ABC and whose base is properly contained in PS. Thus, SPR was not minimal.

In case (8), one vertex of ABC is R, another (denoted by D) belongs to Sm_R, and the third lies on Pm_S. Since SPR is an isosceles triangle, we have $\sphericalangle RDS \geq 90°$. Hence, ABC can be slightly rotated about R so that it remains within SPR, which leads to a contradiction.

It remains to handle case (4). We distinguish two subcases. Denote the apex angle $\sphericalangle SRP$ by δ. If $\delta \geq 60°$, then we can rotate ABC about S. Indeed, vertex D of ABC belongs to $m_S P$, while the base of the altitude belonging to PR lies on $m_S R$. Hence, we have $\sphericalangle SDP \geq 90°$, and the image of ABC through a small clockwise rotation about S is still contained in SPR. Therefore, in this case, SPR cannot be minimal either.

Therefore, from now on we assume $\delta < 60°$. Choose a suitable coordinate system, in which the vertices of ABC are $(0,0)$, $(s,0)$, and $D = (p,q)$. We also have $S = (0,0)$ and $R = (s+x, 0)$ for some $x > 0$. Since $\delta < 60° < 90°$, vertex D is to the left of R, that is, $p < s + x$.

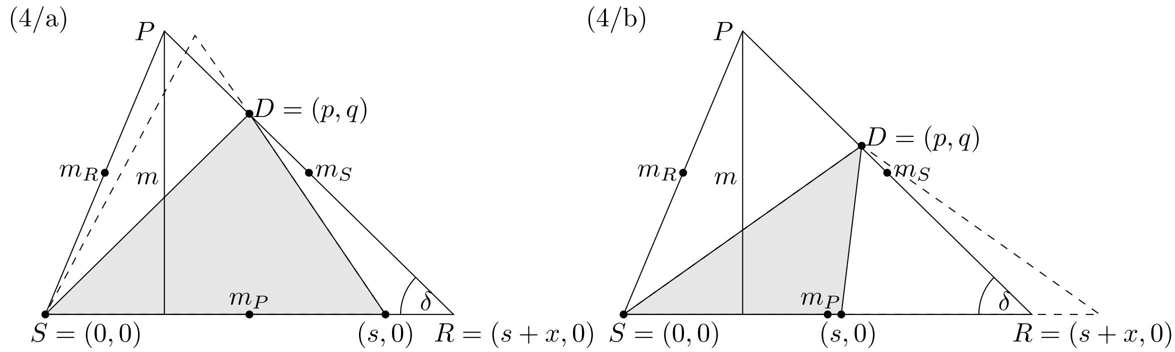

FIGURE 8. Possible realizations of Case (4) when $\delta < 60°$.

By simple calculation,

$$P = (s+x, 0) + \frac{(s+x)}{\sqrt{(p-(s+x))^2+q^2}}(p-(s+x), q). \tag{1}$$

Denote by m the length of the altitude of SPR belonging to the side SR. Then, m is equal to the second coordinate of P (see Figure 8). We have

$$m = \frac{q(s+x)}{\sqrt{(p-(s+x))^2+q^2}}.$$

Let us compute the derivative of the function

$$f(x) = 2t(SPR) = q(s+x)^2 \cdot \left((p-(s+x))^2+q^2\right)^{-\frac{1}{2}}.$$

We obtain

$$\begin{aligned} f'(x) &= 2q(s+x)\left((p-(s+x))^2+q^2\right)^{-\frac{1}{2}} + q(s+x)^2(p-(s+x))\left((p-(s+x))^2+q^2\right)^{-\frac{3}{2}} \\ &= q(s+x)\left((p-(s+x))^2+q^2\right)^{-\frac{3}{2}}\left[(p-(s+x))(2p-(s+x))+2q^2\right] \\ &= \underbrace{q\,(s+x)\,\left((p-(s+x))^2+q^2\right)^{-\frac{3}{2}}}_{>0}\left[\left(\frac{3}{2}p-(s+x)\right)^2 - \frac{1}{4}p^2+2q^2\right]. \end{aligned}$$

Case (4/a1): $q \geq \frac{1}{2}p$. Then, $-\frac{1}{4}p^2+2q^2 > 0$, and hence, $f'(x) > 0$ for all $x \geq 0$. Thus, f is strictly increasing and since x cannot be negative, f takes its minimum at $x = 0$. This means that the area of a special container of the first kind where $x = 0$ (see the triangle with dashed sides on Figure 8 (4/a)) is smaller than the area of SPR for $x > 0$.

Case (4/a2): $2p < s+x$. Then $\frac{1}{2}p < (s+x) - \frac{3}{2}p$, so that $(\frac{3}{2}p-(s+x))^2 - (\frac{1}{2}p)^2 > 0$. Again, we have $f'(x) > 0$ for all $x \geq 0$ and, as above, we obtain a special container of the first kind whose area is smaller than the area of SPR (see Figure 8 (4/a)).

Case (4/b): $q < \frac{1}{2}p$ and $2p \geq s+x$. Let Δ denote the triangle with vertices $(0,0)$, (p,q), and $(2p,0)$. It follows from the inequality $2p \geq s+x$ that Δ is an isosceles container of the second kind associated with ABC (see Figure 8 (4/b)). We show that Δ has smaller area than SPR. To prove this, we have to verify that

$$t(\Delta) = pq < q\frac{(s+x)^2}{2\sqrt{(p-(s+x))^2+q^2}} = t(SPR).$$

Using our assumption that $(p,q) \in m_S P$, we obtain

$$p \leq s+x+\frac{(s+x)(p-(s+x))}{2\sqrt{(p-(s+x))^2+q^2}}.$$

The right-hand side of the last inequality is the first coordinate of the midpoint m_S of PR, where P is given by formula (1) and $R = (s+x, 0)$. Thus, it is sufficient to show that

$$(s+x) + \frac{(s+x)(p-(s+x))}{2\sqrt{(p-(s+x))^2+q^2}} < \frac{(s+x)^2}{2\sqrt{(p-(s+x))^2+q^2}},$$

which reduces to $3p^2+4q^2 < 4(s+x)p$. The last inequality holds, because $p < s+x$ and, by our assumption, $2q \leq p$. This completes the proof of Theorem 1.3. $\square$

Proof of Theorem 1.4. By Theorem 1.3, every minimum area isosceles container for a triangle ABC is one of its special containers. By Lemma 2.2, it must be a special container of the first or the second kind.

Suppose for a contradiction that a minimum area special container SPR associated with the triangle ABC is acute, but it is of the second kind. Assume without loss of generality that $S = B$ and $R = A$. (The other cases can be treated in a similar manner.) By our notation, $P = C_2$. We prove that $t(ABC') < t(ABC_2) = t(SPR)$ (see Figure 9).

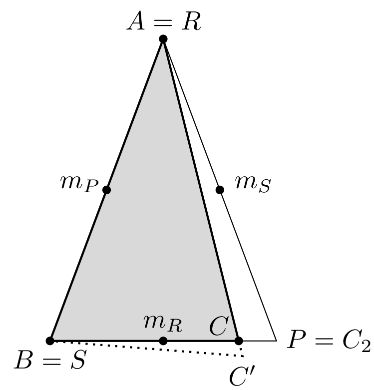

FIGURE 9. An acute minimum area isosceles container cannot be a special container of the second kind.

Indeed, we have $|AB| = |AC_2| = |AC'|$ and $\sphericalangle C'AB < \sphericalangle C_2AB$. Since $SPR = ABC_2$ is acute, it follows that $t(ABC') < t(ABC_2)$. □

Corollary 4.1. *If a minimum area isosceles container for ABC is a special container of the second kind, then it must be AB_1C.*

Proof. By Theorem 1.4, if a special container of the second kind has minimum area, then it has to be non-acute. If ABC_2 is non-acute, then ABC_1 is obtuse and $t(ABC_2) > t(ABC_1)$, because $|AB| = |AC_1| = |AC_2|$ and $\sphericalangle ABC_1 > \sphericalangle BAC_2 \geq 90°$. On the other hand, as AB_1C and ABC_1 share a base angle at A and $b < c$, it follows that $t(ABC_1) > t(AB_1C)$. □

5. Quantitative results—Proofs of Theorems 1.6 and 1.7 (a)

Proof of Theorem 1.6. By Theorem 1.3, a minimal area isosceles container for ABC is a special container associated with ABC. In view of Lemma 2.2, it must be a special container of the first or second kind. By Lemma 2.1 (a) and Corollary 4.1, among special containers of the first kind, it is enough to consider ABC' and $AB'C$, and among special containers of the second kind, only AB_1C. These immediately show that every triangle ABC admits at most 3 minimum area isosceles containers.

If ABC is an obtuse or right triangle, then $t(AB'C) > t(AB_1C)$. Indeed, in this case $|AC| = |CB'| = |CB_1|$, both $AB'C$ and AB_1C are obtuse or right triangles, and their apex angles satisfy $\sphericalangle ACB' < \sphericalangle ACB_1$. Thus, there are only two candidates for a minimum area isosceles container: ABC' and AB_1C.

If ABC is an acute triangle and it has 3 minimum area isosceles containers, then $t(AB'C) = t(ABC') = t(AB_1C)$. Since $t(BCB_1) = t(BCC')$, we obtain

$$(c - b)\sin(\alpha + \beta) = b\sin(\beta - \alpha). \tag{2}$$

Note that this equation also holds when ABC is obtuse.

It follows from equation (2) that $\frac{c}{b} = \frac{2\sin(\beta)\cos(\alpha)}{\sin(\alpha+\beta)}$. By Lemma 2.1 (b), the equation $t(AB'C) = t(ABC')$ reduces to $\frac{c}{b} = \frac{b}{a}$. Thus, $\frac{\sin(\beta)}{\sin(\alpha)} = \frac{2\sin(\beta)\cos(\alpha)}{\sin(\alpha+\beta)}$, so that $\sin(\alpha+\beta) = \sin(2\alpha)$. Therefore, either $\alpha = \beta$, which is impossible, or $180° - (\alpha + \beta) = \gamma = 2\alpha$.

It follows from $\frac{c}{b} = \frac{b}{a}$ that

$$\frac{\sin(2\alpha)}{\sin(3\alpha)} = \frac{\sin(3\alpha)}{\sin(\alpha)}. \tag{3}$$

Since ABC is acute, its smallest angle is α, and $180° - 3\alpha = \beta < \gamma = 2\alpha$, we have that $36° < \alpha < 45°$. Simple analysis shows that equation (3) has exactly one solution α^* in the interval $[36°, 45°]$. It can be approximated by computer. The other two angles of the corresponding triangle are $\beta^* = 180° - 3\alpha^*$ and $\gamma^* = 2\alpha^*$. □

Example 5.1. By the proof of Theorem 1.6, any minimal area isosceles container for ABC is either $AB'C$, or ABC', or AB_1C. Here, we construct a family of acute triangles ABC whose only minimal area isosceles containers are special containers of the second kind, i.e., AB_1C. Moreover, AB_1C is obtuse.

Let $\alpha > \alpha^*$ and $90° > \gamma > 2\alpha > \gamma^*$. Then, denoting by $\beta = 180° - \alpha - \gamma$ we obtain that

$$\frac{\sin(\gamma)}{\sin(\beta)} > \frac{\sin(\gamma^*)}{\sin(\beta^*)} = \frac{\sin(\beta^*)}{\sin(\alpha^*)} > \frac{\sin(\beta)}{\sin(\alpha)},$$

which implies that $t(AB'C) < t(ABC')$. The triangles AB_1C and $AB'C$ are isosceles with legs of length b, so it is enough to show that $\sin(\sphericalangle ACB_1) = \sin(180° - 2\alpha) < \sin(\gamma) = \sin(\sphericalangle ACB')$. However, this follows from the inequalities $90° > \gamma > 2\alpha$. The base angle of AB_1C satisfies $\alpha < 45°$, so that AB_1C is obtuse.

Proof of Theorem 1.7 (a): Let r^* denote the supremum of the ratio of the area of a minimum area isosceles container of a triangle to the area of the triangle itself. In view of Theorem 1.6, we have

$$r^* = \sup_{\text{triangle } ABC} \min\Big(\frac{t(AB'C)}{t(ABC)}, \frac{t(ABC')}{t(ABC)}, \frac{t(AB_1C)}{t(ABC)}\Big).$$

If $\beta \geq 45°$, then $\sin(\beta) \geq \frac{1}{\sqrt{2}}$. Using the proof of Lemma 2.1(b) and the law of sines, we obtain

$$\frac{t(ABC')}{t(ABC)} = \frac{c}{b} = \frac{\sin(\gamma)}{\sin(\beta)} \leq \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}.$$

Equality holds here if and only if $\beta = 45°$ and $\gamma = 90°$, in which case ABC is an isosceles triangle and the ratio of the area of the minimum isosceles container to the area of ABC is 1.

If $\beta < 45°$, then $\gamma > 90°$. Hence, ABC is obtuse and, by the proof of Theorem 1.6, the minimum area isosceles container is ABC' or AB_1C. For fixed β and c, we can express the ratios of the areas as functions of α. Let

$$f(\alpha) = \frac{t(ABC')}{t(ABC)} = \frac{c}{b}$$

and

$$g(\alpha) = \frac{t(AB_1C)}{t(ABC)} = \frac{2b\cos(\alpha)}{b\cos(\alpha) + a\cos(\beta)} = \frac{1}{\frac{1}{2} + \frac{\tan(\alpha)}{2\tan(\beta)}},$$

where $0 < \alpha < \beta$. Obviously, $f(\alpha)$ is strictly increasing and $g(\alpha)$ is strictly decreasing on the open interval $(0, \beta)$, and both functions are continuous. We have

$$\lim_{\alpha\to 0+} f(\alpha) = 1, \quad 1 < \lim_{\alpha\to\beta-} f(\alpha) < 2,$$

$$\lim_{\alpha\to 0+} g(\alpha) = 2, \quad \lim_{\alpha\to\beta-} g(\alpha) = 1.$$

Therefore, the graphs of f and g intersect at a unique point z. Thus, $\max_{0°<\alpha<\beta}(\min(f(\alpha), g(\alpha))) = f(z) = g(z)$, which implies $t(ABC') = t(AB_1C)$. This means that $t(BCB_1) = t(BCC')$, so that equation (2) above holds. Using the law of sines, we obtain $(\frac{c}{b})^2 = 2\cos(z) < 2$ and, hence, $\frac{c}{b} < \sqrt{2}$. If $\beta \to 0$, then $z \to 0$ and $c/b \to \sqrt{2}$. This implies that $r^* = \sqrt{2}$, but the supremum is not realized by any triangle ABC. □

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References

[1] P. Bose and J.-L. De Carufel, Minimum-area enclosing triangle with a fixed angle, *Computational Geometry* **47** (2014), 90–109.

[2] J. E. Boyce, D. P. Dobkin, R. L. Drysdale, and L. J. Guibas, Finding extremal polygons, *SIAM J. Computing* **14** (1985), 134–147.

[3] H. E. Debrunner and P. Mani-Levitska,Can you cover your shadows?, *Discrete Comput. Geom.* **1** (1) (1986), 45–58.

[4] C. H. Dowker, On minimum circumscribed polygons, *Bull. Amer. Mathematical Soc.*, **10** (1944), 120–122.

[5] H. G. Eggleston, On triangles circumscribing plane convex sets, *J. London Mathematical Society* **s1-28** (1-2) (1953), 36–46.

[6] L. Fejes Tóth, *Lagerungen in der Ebene, auf der Kugel und im Raum, Die Grundlehren der Mathematischen Wissenschaften (in German)*, **LXV**, Springer-Verlag, Berlin, 1953.

[7] M. Grötschel, L. Lovász, and A. Schrijver, *Geometric Algorithms and Combinatorial Optimization,* Springer, New York, 1988.

[8] D. Ismailescu, Circumscribed polygons of small area, *Discrete & Computational Geometry* **41** (2009), 583–589.

[9] R. P. Jerrard and J. E. Wetzel, Equilateral triangles and triangles, *Amer. Math. Monthly*, **109** (10) (2002), 909–915.

[10] F. John, Extremum problems with inequalities as subsidiary conditions, In: *Studies and Essays presented to R. Courant on his 60th birthday*, Interscience Publishers, NY, 1948, 187–204.

[11] A. Kanazawa, On the minimal volume of simplices enclosing a convex body, *Archiv der Mathematik* **102** (5) (2014), 489–492.

[12] G. Kós and J. Törőcsik, Convex disks can cover their shadow, *Discrete Comput. Geom.* **5** (6) (1990), 529—-531.

[13] P. Kumar and E. A. Yıldırım, Minimum volume enclosing ellipsoids and core sets, *J. Optimization Theory and Appls* **126** (1) (2005), 1–21.

[14] L. Moser, Poorly formulated unsolved problems of combinatorial geometry, Mimeographed list, 1966.

[15] R. Nandakumar, Oriented convex container of polygons, https://arxiv.org/ftp/arxiv/papers/1802/1802.10447.pdf

[16] J. O'Rourke, Finding minimal enclosing boxes, *Internat. J. Computer and Information Sc.* **14** (3) (1985), 183–199.

[17] K. A. Post, Triangle in a triangle: On a problem of Steinhaus, *Geometriae Dedicata* **45** (1) (1993), 115–120.

[18] B. V. Rublev and Y. I. Petunin, Minimum-area ellipse containing a finite set of points I., *Ukrainian Mathematical J.* **50** (7) (1998), 1115–1124.

[19] B. V. Rublev and Y. I. Petunin, Minimum-area ellipse containing a finite set of points II., *Ukrainian Mathematical J.* **50** (8) (1998), 1253–1261.

[20] A. Schrijver, *Combinatorial Optimization. Polyhedra and Efficiency. Vols A, B, C. Algorithms and Combinatorics* **24**, Springer-Verlag, Berlin, 2003.

[21] H. Steinhaus, *One Hundred Problems in Elementary Mathematics,* Pergamon, Oxford, 1964, p. 98.

[22] E. Welzl, Smallest enclosing disks (balls and ellipsoids), In: *Proc. New Results and New Trends in Computer Science, Lecture Notes in Comp. Sc.* **555** (1991), 359–370.

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