

From Tarski’s Plank Problem to Simultaneous Approximation

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Abstract. A *slab* (or plank) is the part of the d -dimensional Euclidean space that lies between two parallel hyperplanes. The distance between these hyperplanes is called the *width* of the slab. It is conjectured that the members of any infinite family of slabs with divergent total width can be translated so that they together cover the whole d -dimensional space. We prove a slightly weaker version of this conjecture, which can be regarded as a converse of Bang’s theorem, also known as Tarski’s plank problem.

This result enables us to settle an old conjecture of Makai and Pach on simultaneous approximation of polynomials. We say that an infinite sequence S of positive numbers *controls* all polynomials of degree at most d if there exists a sequence of points in the plane whose x -coordinates form the sequence S , such that the graph of every polynomial of degree at most d passes within vertical distance 1 from at least one of the points. We prove that a sequence S has this property if and only if the sum of the reciprocals of the d -th powers of its elements is divergent.

1. TARSKI’S PLANK PROBLEM AND ITS AFFINE VERSION. The closed set of points S lying between two parallel hyperplanes in \mathbb{R}^d at distance w from each other is called a *slab* (or *plank*) of *width* w . Given a convex body $C \subset \mathbb{R}^d$, its *width* $w(C)$ is the smallest number w such that there is a slab of width w that covers C .

In 1932, *Alfred Tarski* [18] made the following attractive conjecture.

Tarski’s plank problem. *If a sequence of slabs covers a convex body C , then the total width of the slabs is at least $w(C)$.*

Tarski found a beautiful proof of his conjecture in the special case where C is a closed disk. His argument was based on the following fact discovered by *Archimedes* more than two thousand years ago. Let D be a disk of radius $\frac{1}{2}$ in the (x, y) -plane \mathbb{R}^2 , and let H denote the hemisphere of radius $\frac{1}{2}$, concentric with D , that lies in the closed half-space above \mathbb{R}^2 . No matter how we place a 2-dimensional slab S of width w in the plane such that both of its boundary lines intersect D , the surface area of the vertical projection of $S \cap D$ to the hemisphere H is always equal to $\frac{1}{2}\pi w$; see Fig. 1. Suppose now that a system of slabs S_1, S_2, \dots completely covers D . We can assume without loss of generality that the boundary lines of every slab intersect D . Otherwise, we could replace one of the slabs with a narrower one. The vertical projections of the slabs cover H , therefore the sum of the areas of these projections, $\frac{1}{2}\pi w(S_1) + \frac{1}{2}\pi w(S_2) + \dots$, is at least $Area(H) = \frac{\pi}{2}$. Thus, we have that

$$w(S_1) + w(S_2) + \dots \geq 1 = w(D),$$

as required.

A slight modification of the above argument also settles the analogous problem in the case where C is a 3-dimensional ball B^3 of unit diameter. Now let H denote the whole sphere bounding B^3 , so that $Area(H) = \pi$. Suppose that S_1, S_2, \dots is a system of 3-dimensional slabs that altogether cover B^3 , with the property that the boundary planes of each S_i intersect B^3 . By the Archimedean observation, for every i , the surface area of $S_i \cap H$ is equal to $\pi w(S_i)$. Since the slabs must also cover H , we have

$$\sum_i Area(S_i \cap H) = \sum_i \pi w(S_i) \geq Area(H) = \pi,$$

which again yields that $\sum_i w(S_i) \geq w(B^3) = 1$.

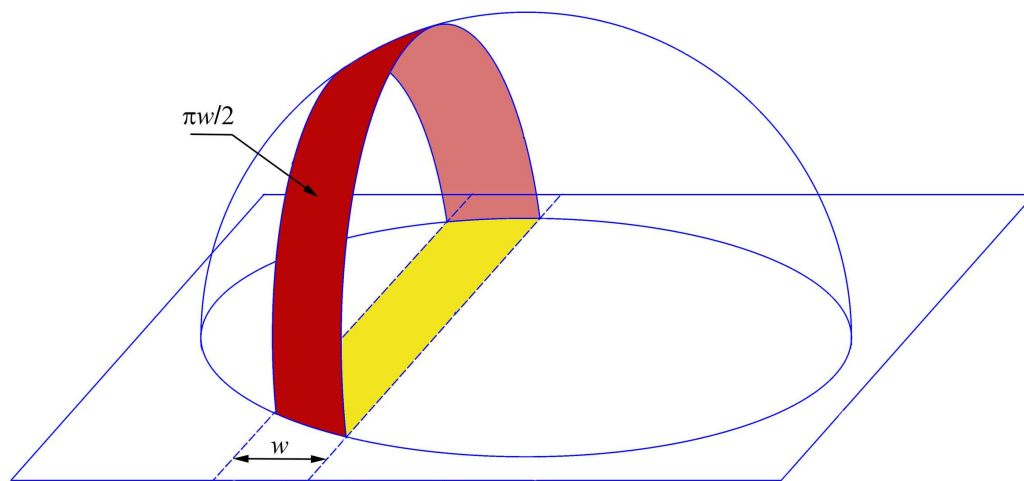


Figure 1. Tarski's proof for a disk of unit diameter.

One may naively hope that the above argument allows a straightforward generalization to higher dimensional balls or even to convex bodies of other shapes. This is not the case. It took almost twenty years before *Thøger Bang* [2], [3] managed to prove Tarski's conjecture, using quite different ideas. He also formulated a more general conjecture that he was unable to settle. To state Bang's conjecture, we need some notation.

A unit vector \mathbf{v} perpendicular to the bounding hyperplanes of a slab S is called the *normal vector* of S . Given a convex body C and a unit vector $\mathbf{v} \in \mathbb{R}^d$, let $w(C, \mathbf{v})$ denote the *width of C in direction \mathbf{v}* , that is, the smallest number w such that there is a slab of width w with normal vector \mathbf{v} that contains C . For example, if C is a d -dimensional ball of unit diameter, then $w(C, \mathbf{v}) = 1$ for any unit vector $\mathbf{v} \in \mathbb{R}^d$.

Bang's affine plank problem. *Let S_1, S_2, \dots be a sequence of slabs in \mathbb{R}^d with normal vectors $\mathbf{v}_1, \mathbf{v}_2, \dots$ and widths $w(S_1), w(S_2), \dots$, respectively, and suppose that their union covers a convex body C .*

Then we have

$$\sum_i \frac{w(S_i)}{w(C, \mathbf{v}_i)} \geq 1.$$

This conjecture is still open. Bang [4] verified it for systems consisting of only *two* slabs (see also [16]), but it is not known to be true even for triples. It was a sensational breakthrough, when in 1991 *Keith Ball* [1] settled the conjecture in the affirmative for centrally symmetric convex bodies C . On the other hand, some results of *Richard Gardner* [9] indicate that the affine plank problem cannot be solved by any argument similar to Tarski's.

2. A CONVERSE OF TARSKI'S PROBLEM. Tarski's conjecture, that is, Bang's theorem, states that if a sequence of slabs cover a convex body $C \subset \mathbb{R}^d$, then their total width must be large. One can reverse this question, as follows. Suppose that we have a sequence of slabs S_1, S_2, \dots of large total width. Is it always possible to cover C with congruent copies of the S_i ? The answer is simple. Take a slab S of width $w(C)$ that contains C , and denote its normal vector by \mathbf{v} . If $\sum_i w(S_i) \geq w(C)$, we can rotate each S_i into a position perpendicular to \mathbf{v} . Now we can translate these slabs so that their union will cover S and, hence C .

The problem becomes more interesting if we permit only translations, but no rotation. We say that a sequence of slabs S_1, S_2, \dots permits a *translative covering* of a subset $C \subseteq \mathbb{R}^d$ if there are suitable translates S'_i of S_i ($i = 1, 2, \dots$) such that $\cup_i S'_i \supseteq C$.

If each of the slabs has width greater than some positive constant ϵ , then each can be used to cover a full-dimensional ball of diameter ϵ . Since any convex body C can be covered by finitely many balls of diameter ϵ , these slabs permit a translative covering of C .

However, it is not clear whether the condition that the total width of the slabs is *divergent* is sufficient to guarantee that they permit a translative covering of a ball of unit diameter. If this is true, then any sequence of slabs with divergent total width also permits a translative covering of the *whole space* \mathbb{R}^d . Indeed, any such sequence can be partitioned into infinitely many subsequences, each having divergent total width. Choose any covering of \mathbb{R}^d with balls B_1, B_2, \dots of unit diameter, and for each B_i , use translates of the slabs belonging to the i th subsequence to cover B_i ($i = 1, 2, \dots$). *Endre Makai* and *János Pach* [15] made the following conjecture; see also [5], Section 3.4.

Makai–Pach translative plank conjecture. *Let d be a positive integer. A sequence of slabs in \mathbb{R}^d with widths w_1, w_2, \dots permits a translative covering of \mathbb{R}^d if and only if $\sum_{i=1}^{\infty} w_i = \infty$.*

The “only if” part is quite easy. It also follows directly from Bang’s theorem. If $\sum_{i=1}^{\infty} w_i < D$, then the slabs do not even permit a translative covering of a ball of diameter D .

As for the “if” part, in the 2-dimensional case it was proved in [15] and, according to [10], independently, by *Paul Erdős* and *Ernst G. Straus* (unpublished). In this case, there is a constant $c > 0$ such that any system of slabs in the plane with total width at least c permits a translative covering of a disk of unit diameter, and hence the conjecture is true. (See [11, 12] for some refinements.) For $d \geq 3$, the best known result is due to *Helmut Groemer* [10]. He proved that for any $d \geq 3$, any sequence of slabs of widths w_1, w_2, \dots , satisfying

$$\sum_{i=1}^{\infty} w_i^{\frac{d+1}{2}} = \infty,$$

permits a translative covering of \mathbb{R}^d . In particular, for $d = 3$, any sequence of slabs with widths $w_i = \frac{1}{\sqrt{i}}$ ($i = 1, 2, \dots$) permits a translative covering of \mathbb{R}^3 . Our next result shows that the same is true for much narrower slabs, for example, for slabs of widths $w_i = \frac{1}{i}$ for every i . This comes rather close to the truth; the last statement is false, e.g., for the sequence $w_i = 1/i^{1+\epsilon}$ ($i = 1, 2, \dots$) with any $\epsilon > 0$, because then we have $\sum_{i=1}^{\infty} w_i < \infty$.

Theorem 1. *Let d be a positive integer, and let $w_1 \geq w_2 \geq \dots$ be a monotone decreasing infinite sequence of positive numbers such that*

$$\limsup_{n \rightarrow \infty} \frac{w_1 + w_2 + \dots + w_n}{\log(1/w_n)} > 0.$$

Then any sequence of slabs S_i of width w_i ($i = 1, 2, \dots$) permits a translative covering of \mathbb{R}^d .

Here, and in what follows, \log stands for the natural logarithm.

Our proof is based on the following statement.

Theorem 2. *Let d be a positive integer, and let $w_1 \geq w_2 \geq \dots \geq w_n$ be positive numbers such that*

$$w_1 + w_2 + \dots + w_n \geq 3d \log(2/w_n).$$

Then any sequence of slabs $S_1, \dots, S_n \subset \mathbb{R}^d$ with widths w_1, \dots, w_n , resp., permits a translative covering of a d -dimensional ball of diameter $1 - w_n/2$.

Theorems 1 and 2 will be established in Section 4.

3. APPLICATION TO APPROXIMATION OF POLYNOMIALS. For a fixed positive integer d , let \mathcal{P}_d denote the class of polynomials of degree at most d . Following [15], we say that a sequence of positive numbers x_1, x_2, \dots is \mathcal{P}_d -controlling if there exist reals y_1, y_2, \dots with the property that for every polynomial $p \in \mathcal{P}_d$, one can find an i with

$$|p(x_i) - y_i| \leq 1.$$

Roughly speaking, this means that the graph of every polynomial $p \in \mathcal{P}_d$ comes vertically close to at least one point of the set $\{(x_i, y_i) : i = 1, 2, \dots\}$.

In Section 5, we study the following question. How sparse can a \mathcal{P}_d -controlling sequence be? A similar question, motivated by a problem of László Fejes Tóth [8], was studied in [6]. We will see that this question is intimately related to the translative plank conjecture discussed in the previous section.

First, we show how the following assertion can be deduced from Theorem 1.

Corollary 1. *Let d be a positive integer and $x_1 \leq x_2 \leq \dots$ be a monotone increasing infinite sequence of positive numbers. If*

$$\limsup_{n \rightarrow \infty} \left(\frac{1}{x_1^d} + \frac{1}{x_2^d} + \dots + \frac{1}{x_n^d} \right) / \log x_n > 0,$$

then the sequence x_1, x_2, \dots is \mathcal{P}_d -controlling.

Proof of Corollary 1. Let $x_1 \leq x_2 \leq \dots$ be an infinite sequence of positive numbers satisfying the assumptions. We have to find a sequence of reals y_1, y_2, \dots such that for any polynomial $p(x) = \sum_{j=0}^d a_j x^j$ with real coefficients a_j , there exists a positive integer i with $|p(x_i) - y_i| \leq 1$. See Fig. 2.

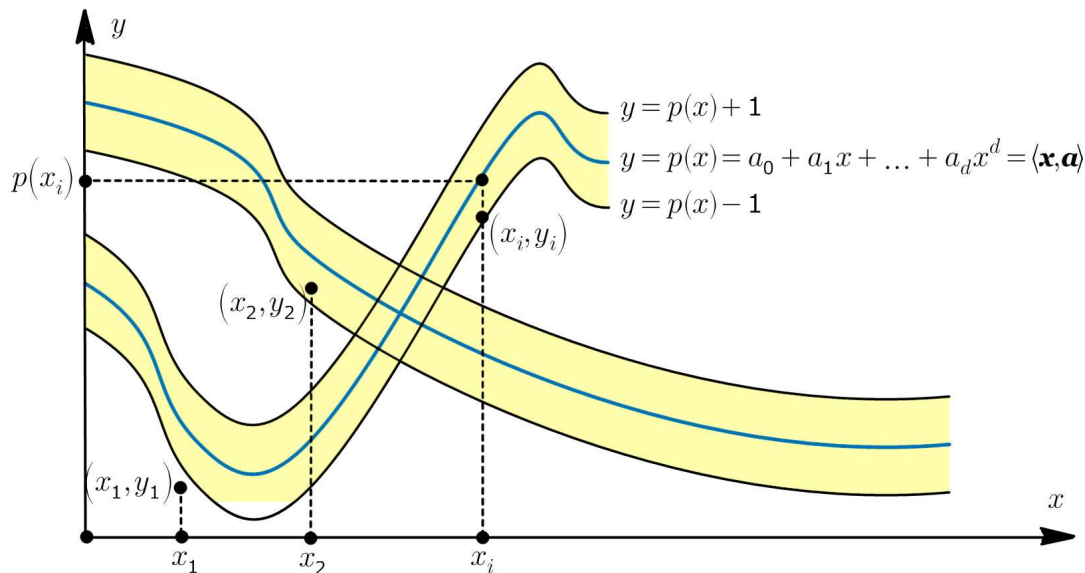


Figure 2. Controlling polynomials of degree at most d .

Write $p(x)$ in the form $p(x) = \langle \mathbf{x}, \mathbf{a} \rangle$, where $\mathbf{x} = (1, x, \dots, x^d)$, $\mathbf{a} = (a_0, a_1, \dots, a_d) \in \mathbb{R}^{d+1}$, and $\langle \cdot \rangle$ stands for the scalar product. Using this notation, we have $\mathbf{x}_i = (1, x_i, \dots, x_i^d)$ and the inequality $|p(x_i) - y_i| \leq 1$ can be rewritten as

$$y_i - 1 \leq \langle \mathbf{x}_i, \mathbf{a} \rangle \leq y_i + 1.$$

For a fixed i , the locus of points $\mathbf{a} \in \mathbb{R}^{d+1}$ satisfying this double inequality is a slab $S_i \subset \mathbb{R}^{d+1}$ of width $w_i = \frac{2}{\|\mathbf{x}_i\|} = \frac{2}{(\sum_{j=0}^d x_i^{2j})^{1/2}}$, with normal vector \mathbf{x}_i . The sequence x_1, x_2, \dots is \mathcal{P}_d -controlling if and only if the sequence of slabs S_1, S_2, \dots permits a translative covering of \mathbb{R}^{d+1} .

We distinguish two cases. If $x_i \leq 3$ for infinitely many (and, hence, for all) integers i , then for the widths of the corresponding slabs we have $w_i > \frac{1}{3^d}$. Thus, these slabs permit a translative covering of \mathbb{R}^{d+1} .

Suppose next that $x_i > 3$ for all $i \geq m$. Then the sequence $w_m \geq w_{m+1} \geq w_{m+2} \geq \dots$ satisfies the condition of Theorem 1. Indeed, we have $w_i \geq \frac{1}{x_i^d}$ for all $i \geq m$, which implies that

$$\limsup_{n \rightarrow \infty} \frac{w_m + w_{m+1} + \dots + w_n}{\log(1/w_n)} \geq \limsup_{n \rightarrow \infty} \left(\frac{1}{x_1^d} + \frac{1}{x_2^d} + \dots + \frac{1}{x_n^d} \right) / \log x_n > 0.$$

Thus, by Theorem 1, the sequence of slabs $S_m, S_{m+1}, S_{m+2}, \dots$ permits a translative covering of \mathbb{R}^{d+1} , which in turn implies that the sequence $x_1 \leq x_2 \leq \dots$ is \mathcal{P}_d -controlling. ■

Note that the above proof yields that for a sequence of positive numbers $x_1 \leq x_2 \leq \dots$ to be \mathcal{P}_d -controlling, it is *necessary* that $\sum_{i=1}^{\infty} \frac{2}{x_i^d} \geq \sum_{i=1}^{\infty} w_i = \infty$. Moreover, if the translative plank conjecture stated in the previous section is true, this condition is also *sufficient*.

In the proof of Corollary 1, we applied Theorem 1 to a very special sequence of slabs S_i , whose normal vectors lie on a moment curve $\gamma(x) = (1, x, x^2, \dots, x^d) \subset \mathbb{R}^{d+1}$. Exploring the natural ordering of these vectors along the curve, in Section 5 we will be able to show that the above condition is indeed necessary and sufficient, without proving the translative plank conjecture.

Theorem 3. *Let d be a positive integer and $x_1 \leq x_2 \leq \dots$ be a monotone increasing infinite sequence of positive numbers. The sequence x_1, x_2, \dots is \mathcal{P}_d -controlling if and only if $\sum_{i=1}^{\infty} \frac{1}{x_i^d} = \infty$.*

This theorem settles Conjecture 3.2.B in [15].

4. PROOFS OF THEOREMS 1 AND 2. We start with the proof of Theorem 2. Then we deduce Theorem 1 from Theorem 2.

The proof of Theorem 2 is based on some ideas that go back (at least) to *Claude Ambrose Rogers* [17, 7].

Proof of Theorem 2. Every slab $S \subset \mathbb{R}^d$ can be expressed in the form

$$S = \{\mathbf{x} \in \mathbb{R}^d : \bar{b} \leq \langle \mathbf{v}, \mathbf{x} \rangle \leq \bar{b} + w\},$$

where \mathbf{v} and w are the unit normal vector and the width of S , respectively, and \bar{b} is a suitable real number.

Fix a sequence of slabs $S_1, \dots, S_n \subset \mathbb{R}^d$ meeting the requirements of the theorem. We may assume that $w_1 < 1$, otherwise we can cover a ball B of diameter 1 with a translate of the first slab. Consider the modified sequence of slabs S'_1, \dots, S'_n , where each S'_i is obtained from S_i by reducing its width by a factor of 2. More precisely, if

$$S_i = \{\mathbf{x} \in \mathbb{R}^d : \bar{b}_i \leq \langle \mathbf{v}_i, \mathbf{x} \rangle \leq \bar{b}_i + w_i\},$$

then let

$$S'_i = \{\mathbf{x} \in \mathbb{R}^d : \bar{b}_i + w_i/4 \leq \langle \mathbf{v}_i, \mathbf{x} \rangle \leq \bar{b}_i + 3w_i/4\}.$$

We describe a greedy algorithm to cover a large part of the unit diameter ball B with suitable translates of S'_1, \dots, S'_n . Set $K_0 = B$. For $i = 1, \dots, n$, let K_{i-1} denote the set of points of B not covered by the translates

of S'_1, \dots, S'_{i-1} selected during the first $i - 1$ steps of the algorithm. In step i , we choose a constant b_i so that the translate

$$T(S'_i) = \{\mathbf{x} \in \mathbb{R}^d : b_i + w_i/4 \leq \langle \mathbf{v}_i, \mathbf{x} \rangle \leq b_i + 3w_i/4\}$$

covers at least a $w_i/3$ -fraction of the volume of K_{i-1} , i.e., we have

$$\text{Vol}(K_{i-1} \setminus T(S'_i)) \leq (1 - w_i/3)\text{Vol}(K_{i-1}).$$

Then $K_i = K_{i-1} \setminus T(S'_i)$. It is not difficult to see that such a b_i exists. Indeed, fix a sequence $b_i^1, \dots, b_i^{\lceil 2/w_i \rceil}$ with the property that the union of translates of S'_i ,

$$\bigcup_{j=1}^{\lceil 2/w_i \rceil} \{\mathbf{x} \in \mathbb{R}^d : b_i^j + w_i/4 \leq \langle \mathbf{v}_i, \mathbf{x} \rangle \leq b_i^j + 3w_i/4\},$$

completely covers B . Since $K_{i-1} \subseteq B$, it follows by the pigeonhole principle that at least one of these translates will cover at least a $\frac{1}{\lceil 2/w_i \rceil}$ -fraction of the volume of K_{i-1} . Notice that $\frac{1}{\lceil 2/w_i \rceil} > \frac{1}{1+2/w_i} = \frac{w_i}{2+w_i} > \frac{w_i}{3}$.

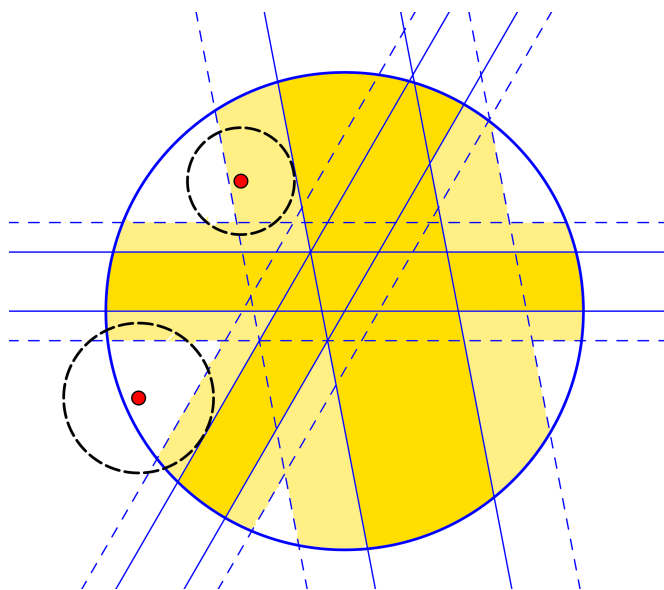


Figure 3. After n steps, the uncovered part of B contains no ball of radius $w_n/4$.

After n steps, the volume of the set of uncovered points of B satisfies

$$\text{Vol}(K_n) \leq \text{Vol}(B) \prod_{i=1}^n (1 - w_i/3) < \text{Vol}(B) \exp \left\{ -\frac{1}{3} \sum_{i=1}^n w_i \right\}.$$

Using the assumption on $\sum_{i=1}^n w_i$, we obtain

$$\text{Vol}(K_n) < \text{Vol}(B) \exp\{-d \log(2/w_n)\} = \text{Vol}(B) \left(\frac{w_n}{2}\right)^d.$$

Therefore, the set $K_n \subseteq B$ does not contain a ball of radius $w_n/4$. See Fig. 3. This implies that every point of B is at distance at most $w_n/4$ from the surface of B or from one of the selected translates $T(S'_i)$ ($i = 1, \dots, n$). In other words, expanding each $T(S'_i)$ by a factor of 2 around its hyperplane of symmetry, we obtain a translate of S_i , and the union of these translates,

$$\bigcup_{i=1}^n \{\mathbf{x} \in \mathbb{R}^d : b_i \leq \langle \mathbf{v}_i, \mathbf{x} \rangle \leq b_i + w_i\},$$

covers the ball of radius $1/2 - w_n/4$, concentric with B . ■

Proof of Theorem 1. It follows from the condition of the theorem that the sequence of slabs S_i can be split into infinitely many finite subsequences $\mathcal{S}_j = (S_{i_{j+1}}, S_{i_{j+2}}, \dots, S_{i_{j+1}})$ for $j = 1, 2, \dots$, where $0 = i_1 < i_2 < i_3 < \dots$, and

$$w_{i_{j+1}} + w_{i_{j+2}} + \dots + w_{i_{j+1}} \geq c \log(1/w_{i_{j+1}}), \tag{1}$$

for a suitable positive constant $c \leq 1$.

We can assume without loss of generality that $w_i \leq \frac{c}{3d}$ holds for every i . Otherwise, if there are finitely many exceptional indices i with $w_i > \frac{c}{3d}$, we simply discard the corresponding slabs. The remaining sequence will meet the requirements of Theorem 1. If the number of exceptional indices is infinite, then we use the construction of the covering for sequences of slabs with bounded widths, described in the beginning of Section 2.

For the same reason, it suffices to show that each subsequence $\mathcal{S}_j = (S_{i_{j+1}}, S_{i_{j+2}}, \dots, S_{i_{j+1}})$ permits a translative covering of a ball of diameter $\frac{c}{6d}$. It follows from Theorem 2 that if

$$\frac{3d}{c}(w_{i_{j+1}} + w_{i_{j+2}} + \dots + w_{i_{j+1}}) \geq 3d \log(2/(\frac{3d}{c}w_{i_{j+1}})), \tag{2}$$

then the slabs $\frac{3d}{c}S$ obtained from the elements $S \in \mathcal{S}_j$ by widening them by a factor of $\frac{3d}{c}$ permit a translative covering of a ball of diameter $1 - \frac{3d}{2c}w_{i_{j+1}}$. Therefore, by scaling, the strips S_i permit a translative covering of a ball of the diameter $\frac{c}{3d}(1 - \frac{3d}{2c}w_{i_{j+1}}) \geq \frac{c}{6d}$. In the last step, we used the assumption $w_{i_{j+1}} \leq \frac{c}{3d}$.

It remains to verify is that (1) implies (2), but this reduces to the inequality $2c \leq 3d$. ■

5. PROOF OF THEOREM 3. It follows from the remark right after the proof of Corollary 1 that we only have to establish the “if” part of the theorem. That is, using the same notation as in the proof of Corollary 1, it is sufficient to show that the sequence of slabs S_i defined there permits a translative covering, provided that the sum $\frac{1}{x_1^d} + \frac{1}{x_2^d} + \dots$ is divergent. As we have seen, we can also assume without loss of generality that $x_i \geq 3$ for every i .

In the proof of Corollary 1, we blindly applied Theorem 1 to the slabs S_i , without exploring their special properties. To refine our argument, we are going to exploit the specifics of the slabs. Recall that S_i has width $w_i > \frac{1}{x_i^d}$ for every i , and its normal vector $\mathbf{x}_i = (1, x_i, \dots, x_i^d)$ lies on the moment curve $(1, x, x^2, \dots, x^d)$. First, we need an auxiliary lemma.

Lemma 1. *Let d be a positive integer, let $3 \leq x_1 \leq x_2 \leq \dots$ be a finite or infinite sequence of reals, and let $\mathbf{x}_i = (1, x_i, x_i^2, \dots, x_i^d)$ for every i . Then there exist $d + 1$ linearly independent vectors $\mathbf{u}_1, \dots, \mathbf{u}_{d+1} \in \mathbb{R}^{d+1}$ such that for every i ($i = 1, 2, \dots$) and j ($j = 1, 2, \dots, d + 1$), we have*

$$(i) \quad \frac{\langle \mathbf{x}_{i+1}, \mathbf{u}_1 \rangle}{\langle \mathbf{x}_i, \mathbf{u}_1 \rangle} \leq \frac{\langle \mathbf{x}_{i+1}, \mathbf{u}_j \rangle}{\langle \mathbf{x}_i, \mathbf{u}_j \rangle},$$

$$(ii) \quad \langle \mathbf{x}_i, \mathbf{u}_j \rangle \geq \frac{1}{3} \|\mathbf{x}_i\| \|\mathbf{u}_j\|.$$

Proof. Take the standard basis $\mathbf{e}_1, \dots, \mathbf{e}_{d+1}$ in \mathbb{R}^{d+1} , i.e., let e_i denote the all-zero vector with a single 1 at the i -th position. Set $\mathbf{u}_j := \mathbf{e}_{d+1-j} + \mathbf{e}_{d+1}$ for $j = 1, \dots, d$ and $\mathbf{u}_{d+1} := \mathbf{e}_{d+1}$.

Condition (i) trivially holds for $j = 1$ and it is very easy to check it for $j = d + 1$. For $j = 2, \dots, d$, it reduces to

$$\frac{x_{i+1}^{d-1} + x_{i+1}^d}{x_i^{d-1} + x_i^d} \leq \frac{x_{i+1}^{d-j} + x_{i+1}^d}{x_i^{d-j} + x_i^d},$$

which is equivalent to

$$(x_{i+1}^{d-1} + x_{i+1}^d)(x_i^{d-j} + x_i^d) \leq (x_{i+1}^{d-j} + x_{i+1}^d)(x_i^{d-1} + x_i^d).$$

The last inequality can be rewritten as

$$x_{i+1}^{d-j} x_i^{d-j} (x_{i+1} - x_i) \left(\sum_{k=0}^{j-1} x_{i+1}^k x_i^{j-1-k} + \sum_{k=0}^{j-2} x_{i+1}^k x_i^{j-2-k} - x_{i+1}^{j-1} x_i^{j-1} \right) \leq 0,$$

or, dividing both sides by $x_{i+1}^{d-j} x_i^{d-j} (x_{i+1} - x_i)$, as

$$\sum_{k=0}^{j-1} x_{i+1}^k x_i^{j-1-k} + \sum_{k=0}^{j-2} x_{i+1}^k x_i^{j-2-k} - x_{i+1}^{j-1} x_i^{j-1} \leq 0.$$

Using the fact $x_{i+1} \geq x_i$, and bounding each sum from above by its largest term multiplied by the number of terms, we obtain that the left-hand side of the last inequality is at most

$$j x_{i+1}^{j-1} + (j-1) x_{i+1}^{j-2} - x_{i+1}^{j-1} x_i^{j-1} < x_{i+1}^{j-1} (2j-1 - x_i^{j-1}).$$

As $x_i \geq 3$, the right-hand side of this inequality is always nonpositive and (ii) holds.

It remains to verify condition (ii). Taking into account that $x_i \geq 3$, we have

$$\langle \mathbf{x}_i, \mathbf{u}_{d+1} \rangle = x_i^d \geq \frac{1}{2} \|\mathbf{x}_i\| = \frac{1}{2} \|\mathbf{x}_i\| \|\mathbf{u}_{d+1}\|.$$

On the other hand, for $j = 1, \dots, d$, we obtain

$$\langle \mathbf{x}_i, \mathbf{u}_j \rangle = x_i^{d-j} + x_i^d \geq \frac{1}{2} \|\mathbf{x}_i\| \geq \frac{1}{3} \|\mathbf{x}_i\| \|\mathbf{u}_j\|.$$

This completes the proof of Lemma 1. ■

In order to establish Theorem 3, it is enough to prove that there is a constant $c = c(d+1)$ such that any system of slabs S_i ($i = 1, \dots, n$) in \mathbb{R}^{d+1} whose normal vectors are $(1, x_i, \dots, x_i^d)$ for some $3 \leq x_1 \leq x_2 \leq \dots \leq x_n$ and whose total width is at least c , permits a translative covering of a ball of unit diameter. This is an immediate corollary of Lemma 1 and the following assertion.

Lemma 2. For every positive integer d , for any system of $d + 1$ linearly independent vectors $\mathbf{u}_1, \dots, \mathbf{u}_{d+1}$ in \mathbb{R}^{d+1} , and for any $\gamma > 0$, there is a constant c with the following property.

Given any system of slabs S_i ($i = 1, \dots, n$) in \mathbb{R}^{d+1} , whose normal vectors \mathbf{x}_i satisfy the conditions

$$(i) \quad \frac{\langle \mathbf{x}_{i+1}, \mathbf{u}_1 \rangle}{\langle \mathbf{x}_i, \mathbf{u}_1 \rangle} \leq \frac{\langle \mathbf{x}_{i+1}, \mathbf{u}_j \rangle}{\langle \mathbf{x}_i, \mathbf{u}_j \rangle},$$

$$(ii) \quad \langle \mathbf{x}_i, \mathbf{u}_j \rangle \geq \gamma \|\mathbf{x}_i\| \|\mathbf{u}_j\|$$

for every i and j , and whose total width $\sum_{i=1}^n w_i$ is at least c , the slabs S_i permit a translative covering of a $(d + 1)$ -dimensional ball of unit diameter.

The proof is based on a greedy algorithm for covering a large simplex in \mathbb{R}^{d+1} by the translates of S_i . One of the vertices of the covered simplex is $\mathbf{0}$, and the others lie on the rays emanating from $\mathbf{0}$ in the directions $\mathbf{u}_1, \dots, \mathbf{u}_{d+1}$. At each step, we place a new slab in such a way that the newly covered part of the ray in direction \mathbf{u}_1 adjoins the previously covered part. Condition (i) guarantees that we do not leave any “hole” between the translates uncovered. Condition (ii) ensures that the simplex completely covered at the end of the procedure is “non-degenerate.” The ratio of any two of its sides is bounded from above by an absolute constant. Finally, we will show that the side of this simplex along \mathbf{u}_1 , and thus every other side, is sufficiently large, which implies that the simplex contains a unit ball.

Proof. Instead of covering a ball of unit diameter, it will be more convenient to cover the simplex Δ with one vertex in the origin $\mathbf{0}$ and the others at the points (vectors) \mathbf{u}_j ($j = 1, \dots, d + 1$). By properly scaling these vectors, if necessary, we can assume that Δ contains a ball of unit diameter.

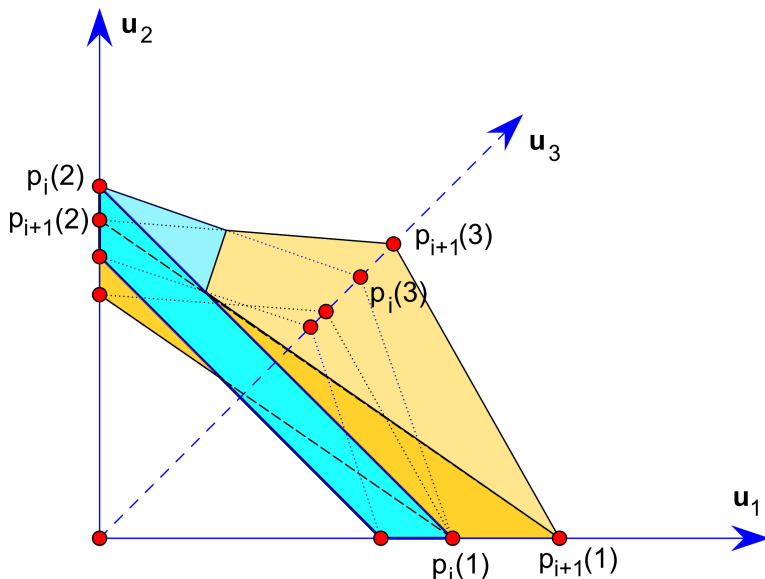


Figure 4. We place the slabs one by one.

We place the slabs one by one. See Fig. 4. We place S'_1 , a translate of S_1 , so that one of its boundary hyperplanes passes through $\mathbf{0}$ and the other one cuts a simplex Δ_1 out of the cone Γ of all linear combinations of the

vectors $\mathbf{u}_1, \dots, \mathbf{u}_{d+1}$ with positive coefficients. According to our assumptions, we have $\langle \mathbf{x}_1, \mathbf{u}_j \rangle > 0$ for every j . Therefore, S'_1 does not separate Γ into two cones: $S'_1 \cap \Gamma$ is indeed a simplex Δ_1 .

We place a translate S'_2 of S_2 in such a way that it adjoins Δ_1 in its vertex along the ray emanating from $\mathbf{0}$ in direction \mathbf{u}_1 . As we will see later, due to condition (i), S'_2 and Δ_1 overlap (in at least one point) along the rays in direction \mathbf{u}_i for $i \geq 2$. Suppose that we have already placed S'_1, \dots, S'_i , the translates of S_1, \dots, S_i , so that their union covers a simplex Δ_i with one vertex at the origin, and the others along the $d + 1$ half-lines that span the cone Γ . We also assume that the facet of Δ_i opposite to the origin is a boundary hyperplane of S'_i . Let $\mathbf{p}_i(j)$ denote the vertex of Δ_i that belongs to the open half-line parallel to \mathbf{u}_j emanating from $\mathbf{0}$ ($j = 1, \dots, d + 1$).

Next, we place a translate S'_{i+1} of S_{i+1} so that one of its boundary hyperplanes, denoted by π , passes through $\mathbf{p}_i(1)$, and the other one, π' , cuts the half-line parallel to \mathbf{u}_1 at a point $\mathbf{p}_{i+1}(1)$ with $\|\mathbf{p}_{i+1}(1)\| > \|\mathbf{p}_i(1)\|$. That is, $\mathbf{p}_{i+1}(1)$ is further away from the origin than $\mathbf{p}_i(1)$ is. Let $\mathbf{p}_{i+1}(2), \dots, \mathbf{p}_{i+1}(d + 1)$ denote the intersection points of π' with the half-lines parallel to $\mathbf{u}_2, \dots, \mathbf{u}_{d+1}$, respectively, and let Δ_{i+1} be the simplex induced by the vertices $\mathbf{0}, \mathbf{p}_{i+1}(1), \dots, \mathbf{p}_{i+1}(d + 1)$.

We have to verify that Δ_{i+1} is entirely covered by the slabs S'_1, \dots, S'_{i+1} . By the induction hypothesis, Δ_i was covered by the slabs S'_1, \dots, S'_i . Thus, it is sufficient to check that the hyperplane π intersects every edge $\mathbf{0}\mathbf{p}_i(j)$ of Δ_i , for $j = 1, \dots, d + 1$. Let $\alpha_j \mathbf{u}_j$ be the intersection point of π with the half-line parallel to \mathbf{u}_j , and let $\mathbf{p}_i(j) = \beta_j \mathbf{u}_j$. We have to prove that $\alpha_j \leq \beta_j$.

By definition, we have $\langle \mathbf{x}_{i+1}, \mathbf{p}_i(1) - \alpha_j \mathbf{u}_j \rangle = 0$ and $\langle \mathbf{x}_i, \mathbf{p}_i(1) - \beta_j \mathbf{u}_j \rangle = 0$. From here, we get

$$\frac{\alpha_j}{\beta_j} = \frac{\langle \mathbf{x}_{i+1}, \mathbf{p}_i(1) \rangle}{\langle \mathbf{x}_{i+1}, \mathbf{u}_j \rangle} \bigg/ \frac{\langle \mathbf{x}_i, \mathbf{p}_i(1) \rangle}{\langle \mathbf{x}_i, \mathbf{u}_j \rangle} = \frac{\langle \mathbf{x}_{i+1}, \mathbf{p}_i(1) \rangle}{\langle \mathbf{x}_i, \mathbf{p}_i(1) \rangle} \bigg/ \frac{\langle \mathbf{x}_{i+1}, \mathbf{u}_j \rangle}{\langle \mathbf{x}_i, \mathbf{u}_j \rangle} = \frac{\langle \mathbf{x}_{i+1}, \mathbf{u}_1 \rangle}{\langle \mathbf{x}_i, \mathbf{u}_1 \rangle} \bigg/ \frac{\langle \mathbf{x}_{i+1}, \mathbf{u}_j \rangle}{\langle \mathbf{x}_i, \mathbf{u}_j \rangle}.$$

In view of assumption (i) of the lemma, the right-hand side of the above chain of equations is at most 1, as required.

Observe that during the whole procedure the uncovered part of the cone Γ always remains convex and, hence, connected. In the n th step, $\cup_{i=1}^n S'_i \supset \Delta_n$. By the construction, $\mathbf{p}_i(1)$ lies at least w_i farther away from the origin along the half-line parallel to \mathbf{u}_1 than $\mathbf{p}_{i-1}(1)$ does. Thus, we have

$$\|\mathbf{p}_n(1)\| \geq \sum_{i=1}^n w_i \geq c.$$

Using the fact that $\langle \mathbf{x}_n, \mathbf{p}_n(j) - \mathbf{p}_n(1) \rangle = 0$ for every $j \geq 2$, and taking into account assumption (ii), we obtain

$$\|\mathbf{p}_n(j)\| \geq \frac{\langle \mathbf{x}_n, \mathbf{p}_n(j) \rangle}{\|\mathbf{x}_n\|} = \frac{\langle \mathbf{x}_n, \mathbf{p}_n(1) \rangle}{\|\mathbf{x}_n\|} \geq \gamma \|\mathbf{p}_n(1)\| \geq \gamma c.$$

Thus, if c is sufficiently large, we have $\|\mathbf{p}_n(j)\| \geq \|\mathbf{u}_j\|$. This means that Δ_n contains the simplex Δ defined in the first paragraph of this proof. Hence, it also contains a ball of unit diameter, as required. ■

6. CONCLUDING REMARKS 1. As was mentioned in Section 2, the translative packing conjecture of Makai and Pach is known to be true in the plane. Moreover, in [15] a stronger statement was proved: there exists a constant c such that every collection of strips with total width at least c permits a translative covering of a disk of diameter 1. In view of this, one can make the following even bolder conjecture.

Strong translative plank conjecture. *For any positive integer d , there exists a constant $c = c(d)$ such that every sequence of slabs in \mathbb{R}^d with total width at least c permits a translative covering of a unit diameter d -dimensional ball.*

Suppose that the *translative plank conjecture* (see Section 2) is true for a positive integer d . Answering a question in [15], Imre Z. Ruzsa [14] proved that then, for the same value of d , the *strong translative plank conjecture* also holds. Thus, the two conjectures are equivalent.

2. We say that a sequence of positive numbers $x_1 \leq x_2 \leq \dots$ is *strongly \mathcal{P}_d -controlling* if there exist reals y_1, y_2, \dots with the property that, for every $\varepsilon > 0$ and for every polynomial p of degree at most d , one can find an i with

$$|f(x_i) - y_i| \leq \varepsilon.$$

It is easy to see that the condition in Theorem 3 is sufficient to guarantee that the sequence x_1, x_2, \dots strongly controls \mathcal{P}_d for every d .

Controlling sequences can be analogously defined for any other class of functions $f : \mathbb{R}^k \rightarrow \mathbb{R}^l$. Several other problems of this kind are discussed in [15] and [13].

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