Points surrounding the origin

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Dedicated to Imre Bárány, Gábor Fejes Tóth, László Lovász, and Endre Makai on the occasion of their sixtieth birthdays.

Abstract

For $d > 2$ and $n > d+1$, let $P = \{p_1, \ldots, p_n\}$ be a set of points in $\mathbb{R}^d$ whose convex hull contains the origin O in its interior. We show that if $P \cup \{O\}$ is in general position, then there exists a d-tuple $Q = \{p_{i_1}, \ldots, p_{i_d}\} \subset P$ such that O is not contained in the convex hull of $Q \cup \{p\}$ for any $p \in P \setminus Q$. Generalizations of this property are also considered.

We also show that for non-empty, finite point sets $A_1, \ldots, A_{d+1}$ in $\mathbb{R}^d$, if the origin is contained in the convex hull of $A_i \cup A_j$ for all $1 \le i < j \le d+1$, then there is a simplex S containing the origin such that $|S \cap A_i| = 1$ for every $1 \le i \le d+1$. This is a generalization of Bárány's colored Carathéodory theorem, and dually, it gives a spherical version of Lovász' colored Helly theorem.

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1 Introduction

Let P be a finite point set in $\mathbb{R}^d$, in *general position with respect to the origin O*, in the sense that no k elements of P lie in a $(k-1)$-dimensional linear subspace of $\mathbb{R}^d$ $(2 \le k \le d)$. We say that P *surrounds the origin* if for every $Q \subset P$ with $|Q| = d$, there exists an $x \in P \setminus Q$ such that the origin is contained in $\operatorname{conv}(\{x\} \cup Q)$, the convex hull of $\{x\} \cup Q$.

In the special case $d = 2$, consider a planar point set $P = \{p_1, p_2, \ldots, p_n\}$, whose elements are listed and enumerated mod n in cyclic order, as they can be seen from the origin. Clearly, P surrounds the origin if and only if for every i, there exists j such that the triangle $p_i p_{i+1} p_j$ contains the origin in its interior. From this, it can be deduced that n must be odd. This so-called "antipodality" property of point sets was explored by Lovász [8] and others [4, 9] to bound the maximum number of "halving lines" of a set of n points in the plane.

R. Strausz [10] discovered the following interesting property of planar point sets P surrounding the origin.

Proposition 1 (Strausz)**.** *For any coloring of the elements of P with* three *colors such that every color is used at least once, there is a* rainbow *triangle which contains the origin in its interior, that is, a triangle whose vertices are of different colors.*

Using the terminology of [1], we can say that the 3-uniform hypergraph consisting of all triples in P whose convex hulls contain the origin is *tight*.

It turns out, somewhat counter-intuitively, that in *three* and higher dimensions, there exists no nontrivial point set that surrounds the origin. By an application of Barnette's inequality for the minimal number of facets of a simplicial polytope, we show the following.

Theorem 2. *Let $d > 2$ and let P be a finite point set in $\mathbb{R}^d$ in general position with respect to the origin, and suppose that $|P| > d+1$. Then P contains a d-tuple Q such that the convex hull of $Q \cup \{x\}$ does not contain the origin for any $x \in P$.*

The above property can be generalized as follows. For any $0 \le k \le d+1$, we say that the set $P \subset \mathbb{R}^d$ has property $S(k)$, if for every $Q \subset P$ with $|Q| = k$, there exists an $R \subset P \setminus Q$ with $|R| = d+1-k$, such that the origin is contained in $\operatorname{conv}(Q \cup R)$.

Obviously, property $S(k)$ depends on the choice of origin, and it is *monotonic* in the sense that property $S(k)$ is stronger than property $S(k-1)$. Carathéodory's theorem (see [6]) states that if the origin is contained in $\operatorname{conv} P$, then it is contained in the convex hull of some $(d+1)$-tuple of P, or simply, $O \in \operatorname{conv} P$ implies property $S(0)$. In fact, we may triangulate P from any given point of P which implies that properties $S(0)$ and $S(1)$ are equivalent.

At the other end of the spectrum, it is easy to show that if $|P| > d+1$, then P does *not* have property $S(d+1)$ (This immediately follows by triangulating the point set). Theorem 2 tells us that for $d > 2$ and $|P| \ge d+1$, properties $S(d+1)$ and $S(d)$ are equivalent. The following two questions arise.

Problem 3. *Let $d \geq 2$ be fixed.*

1. *What is the largest integer $k = k(d)$ such that there are arbitrarily large finite point sets $P \subset \mathbb{R}^d$ in general position with respect to the origin that have property $S(k)$?*
2. *What is the smallest integer $K = K(d)$ such that there is no finite point set $P \subset \mathbb{R}^d$ in general position with respect to the origin with more than $d+1$ elements, which has property $S(K+1)$?*

Clearly, we have $k(d) \leq K(d)$, for every d.

We will prove Theorem 2 in Section 2, and in Section 3 we use the Gale transform to give an equivalent formulation of Problem 3 in terms of facets of convex polytopes. From this viewpoint it will be easy to extract the following lower bound on $k(d)$.

Theorem 4. *For every integer $d \geq 2$, there exist arbitrarily large point sets in $\mathbb{R}^d$ in general position with respect to the origin with property $S(\lfloor \frac{d}{2} \rfloor + 1)$. In other words, $k(d) \geq \lfloor \frac{d}{2} \rfloor + 1$.*

From Theorems 2 and 4 it follows that $k(2) = 2 = K(2)$ and $k(3) = 2 = K(3)$. It would be interesting to know if there are values of d for which $k(d) < K(d)$ holds.

In Section 4 we return to R. Strausz' original observation concerning 3-colorings of planar point sets that surround the origin. We discovered that Proposition 1 is a consequence of the following.

Theorem 5. *Let $A_1, \ldots, A_{d+1}$ be non-empty, finite point sets in $\mathbb{R}^d$. If the origin is contained in the convex hull of $A_i \cup A_j$ for all $1 \leq i < j \leq d+1$, then the origin is contained in some simplex S with $|S \cap A_i| = 1$ for every $1 \leq i \leq d+1$.*

To see how this relates to Proposition 1, consider a 3-coloring of a planar point set P that surrounds the origin, that is, a partition of P into non-empty parts A_1, A_2, A_3. If the origin is not contained in the convex hull of, say $A_1 \cup A_2$, then for any point $a_1 \in A_1$ and any point of $a_2 \in A_2$, the fact that P surrounds the origin implies that there exists a point a_3 which must belong to A_3, such that the origin is contained in the triangle $a_1 a_2 a_3$. On the other hand, if the origin is contained in the convex hulls of the pairwise unions, $A_i \cup A_j$, then there exists a rainbow triangle that contains the origin, by Theorem 5.

Another interesting special case of Theorem 5 is when the origin is contained in the convex hull of A_i for every $1 \leq i \leq d+1$. This is Bárány's version of Carathéodory's theorem [2], which is often called the Colored Carathédory Theorem.

There is a well-known dual statement to Bárány's theorem, often called the Colored Helly Theorem, which had been discovered earlier by Lovász (see Section 5). In Section 5 we establish a dual version of Theorem 5 (Theorem 11), which can be interpreted as a colored version of the Spherical Helly Theorem (see [6] for details).

2 Proof of Theorem 2

A simplicial polytope is a convex polytope for which every k-dimensional face is a k-simplex. In particular, if all the $(d+1)$-tuples of a point set $P \subset \mathbb{R}^d$ are affinely independent, then $\operatorname{conv} P$ is a simplicial polytope, and if P is also in convex position then the number of vertices of $\operatorname{conv} P$ equals $|P|$.

Let P be a d-dimensional polytope. For $0 \leq k \leq d-1$ we denote by $f_k(P)$ the number of k-dimensional faces of P. A special case of Barnette's lower bound theorem [3] states that if P is a simplicial polytope, then we must have

$$f_{d-1}(P) \geq (d-1)f_0(P) - (d+1)(d-2),$$

which for $d = 3$ is an equality by Euler's polyhedral formula.

Proof of Theorem 2. For $d \geq 3$, let P be an n-point set in $\mathbb{R}^d$ $(n \geq d+1)$ in general position with respect to the origin. We will show that if P surrounds the origin, then $n = d+1$ which will prove Theorem 2.

First, project P from the origin to the unit sphere centered at the origin, to obtain a point set P' in convex position. It is clear that P' is also in general position with respect to the origin and that P' surrounds the origin if and only if P does. In fact, it suffices to check this for the case when P is a simplex. Moreover, by projecting some of the points slightly farther away from the origin, if necessary, we can assume that P' is not only in general position with respect to the origin, but that all the $(d+1)$-tuples of P' are affinely independent. Therefore $\operatorname{conv} P'$ is a simplicial polytope on n vertices, which we denote by S.

Let v be a vertex of S. Since P' is in general position with respect to the origin, the line passing through v and O intersects the relative interior of a unique facet of S, which gives us a function

$$\lambda : \{\text{vertices of } F\} \to \{\text{facets of } F\}.$$

Now, if P surrounds the origin, for every facet F of S, there must exist a vertex v of S such that the origin is contained in the simplex spanned by v and F. This implies that $\lambda(v) = F$. Therefore the function λ is surjective, and we must have $f_0(S) \geq f_{d-1}(S)$. By Barnette's inequality, we obtain

$$n = f_0(S) \geq (d-1)f_0(S) - (d+1)(d-2).$$

Since $d \geq 3$, this implies $n \leq d+1$, as required. □

Remark. We found several other proofs of Theorem 2. One can show that if there is an n-point set in $\mathbb{R}^d$ $(d > 3)$ that surrounds the origin, one can, by a suitable projection, get an $(n-1)$-point set in $\mathbb{R}^{d-1}$ that surrounds the origin. Thus we can reduce the problem to $\mathbb{R}^3$.

Another approach is by counting simplices of triangulations of an n-point set in $\mathbb{R}^d$, analogously to the argument in [5].

3 k-surrounding sets

A point set P in general position with respect to the origin O is said to be *k-surrounding*, or is said to have *property* $S(k)$, if any k-element subset of P can be extended to a $(d+1)$-element subset of P that contains O in its interior.

Proof of Theorem 4. The case when d is odd follows from the case when d is even. To see this, suppose $P \subset \mathbb{R}^{2k}$ has property $S(k+1)$, and consider P as a subset of the hyperplane $\{(x_1,\ldots,x_{2k},-1)\} \subset \mathbb{R}^{2k+1}$, such that P surrounds the point $(0,\ldots,0,-1)$. Let $Q = P \cup \{(0,\ldots,0,1)\}$. It is easily seen that Q has property $S(k+1)$: Let $X \subset Q$ be of size $k+1$. If $X \subset P$, there exists a set $Y \subset P$ with $|Y| = k$ such that $(0,\ldots,0,-1) \in \operatorname{conv} X \cup Y$. Then the origin is contained in $\operatorname{conv}(X \cup Y \cup \{(0,\ldots,0,1)\})$. Otherwise, $X = X' \cup \{(0,\ldots,0,1)\}$, where $X' \subset P$ and $|X'| = k$. Taking into account that property $S(k+1)$ implies property $S(k)$, there exists a set $Y \subset P$ with $|Y| = k+1$ such that $(0,\ldots,0,-1) \in \operatorname{conv} X' \cup Y$, and consequently, the origin is contained in $\operatorname{conv} X \cup Y$. Therefore, it suffices to consider the case when d is even.

To complete the proof of Theorem 4, it will be more convenient to transform the problem via the well known Gale transform. (For details concerning the Gale transform, we refer the reader to [7] or [11].)

Let $d \geq 2$ be an integer and suppose $P \subset \mathbb{R}^d$ is in general position with respect to the origin, $|P| = n$, and P has property $S(k)$. The Gale transform of $P \cup \{O\}$ is a $(|P|+1)$-element vector configuration in $\mathbb{R}^{n-d}$, which we denote by $\mathbf{V} \cup \{\mathbf{1}\}$. Here $|\mathbf{V}| = n$ and the vector $\mathbf{1}$ corresponds to the origin O in the "primal" space.

Property $S(k)$ corresponds to the following property of $\mathbf{V}$: For every $\mathbf{U} \subset \mathbf{V}$ with $|\mathbf{U}| = n-k$, there exists $\mathbf{W} \subset \mathbf{U}$ with $|\mathbf{W}| = n-d-1$, such that $(\mathbf{n_W} \cdot \mathbf{1})(\mathbf{n_W} \cdot \mathbf{v}) < 0$ for every $\mathbf{v} \in \mathbf{V} \setminus \mathbf{W}$. Here, $\mathbf{n_W}$, is some fixed vector orthogonal to $\mathbf{W}$, and, $\cdot$, denotes the usual dot product.

In particular, property $S(k)$ implies that there is an $(n-d-1)$-dimensional hyperplane H through the origin with normal vector $\mathbf{n}$ such that $(\mathbf{n} \cdot \mathbf{1})(\mathbf{n} \cdot \mathbf{v}) < 0$ for every $\mathbf{v} \in \mathbf{V}$. Therefore, if we extend the vectors of $\mathbf{V}$ to rays, they will intersect $H - \mathbf{1}$. The set of intersection points, P^*, is a set of n points in general position in $\mathbb{R}^{n-d-1}$ with the following property, denoted by $S^*(k)$: *Among any $n-k$ points of P^*, there are some $n-d-1$ that form a facet of* $\operatorname{conv} P^*$.

In fact, this necessary condition is also sufficient, for one can choose an appropriate vector $\mathbf{1}$ in $\mathbb{R}^{n-d}$, which yields a corresponding point set $P \subset \mathbb{R}^d$ with property $S(k)$. Summarizing:

Observation 6. *There exist n points in $\mathbb{R}^d$ satisfying property $S(k)$ if and only if there exist n points in $\mathbb{R}^{n-d-1}$ satisfying property $S^*(k)$.*

We now complete the proof of Theorem 4. First note that for $d = 2$, the regular $(2n+1)$-gon has property $S(2)$. It remains to exhibit arbitrarily large point sets in $\mathbb{R}^d$ for even $d \geq 4$ with property $S(\frac{d}{2}+1)$.

For positive integers k and $n > 2k-1$, let $C(n,k)$ denote the cyclic polytope on n vertices in $\mathbb{R}^{n-2k+1}$. The facets of $C(n,k)$ have a simple characterization known as Gale's evenness condition (see [7] or [11]). Using this characterization, it is easy to show that when n is odd, $C(n,k)$ has property $S^*(k)$. Hence, by Observation 6, there exist n points in $\mathbb{R}^{2k-2}$ with property $S(k)$. $\square$

By Observation 6, Problem 3 can be reformulated in terms of the property $S^*(k)$. We obtain the following.

Problem 7. *Let $d \geq 2$ be fixed.*

1. *What is the largest integer $k = k(d)$ such that there exists arbitrarily large finite point sets P in general position in $\mathbb{R}^{|P|-d-1}$ that have property $S^*(k)$?*
2. *What is the smallest integer $K = K(d)$ such that there exists no finite point set P in general position in $\mathbb{R}^{|P|-d-1}$ with more than $d+1$ elements, which has property $S^*(K+1)$?*

4 Proof of Theorem 5

Let $\mathbb{S}^{d-1}$ denote the $(d-1)$-dimensional unit sphere in $\mathbb{R}^d$ centered at the origin. By central projection from the origin it is clear that it suffices to prove Theorem 5 for point sets on $\mathbb{S}^{d-1}$. For the proof it will also be convinient to make a general position assumption. We say that a finite point set $A \subset \mathbb{S}^{d-1}$ is in *general position* if any $1 \leq k \leq d$ points of A span a k-dimensional linear subspace of $\mathbb{R}^d$. Thus a set of points is in general position on $\mathbb{S}^{d-1}$ if and only if it is in general position with respect to the origin in (as a point set in $\mathbb{R}^d$). We will prove the following.

Theorem 8. *Let $A_1, \dots, A_{d+1}$ be disjoint, non-empty, finite point sets such that $A_1 \cup \dots \cup A_{d+1}$ is in general position on $\mathbb{S}^{d-1}$. If the origin is contained in the convex hull of $A_i \cup A_j$ for all $1 \leq i < j \leq d+1$, then the origin is contained in some simplex S with $|S \cap A_i| = 1$ for every $1 \leq i \leq d+1$.*

It is clear that Theorem 8 implies Theorem 5. For an infinite sequence of convex sets $\{K_j\}_{j=1}^{\infty}$ converging to a compact convex set K in $\mathbb{R}^d$, if the origin is contained in every K_j, then the origin is contained in K as well. Thus we may replace the points of $\bigcup A_i$ from Theorem 5 by convergent sequences of points in general position on $\mathbb{S}^{d-1}$, obtaining an infinite sequence of point sets $\{C_j\}_{j=1}^{\infty}$, where each $C_j = A_{(1,j)} \cup \dots \cup A_{(d+1,j)}$ satisfies the conditions of Theorem 8. Therefore we can find an infinite sequence $\{S_j\}_{j=1}^{\infty}$ of simplices containing the origin such that $|S_j \cap A_{(i,j)}| = 1$ for every $1 \leq i \leq d+1$. The sequence $\{S_j\}$ converges to a (possibly degenerate) simplex S, where $|S \cap A_i| = 1$ for every $1 \leq i \leq d+1$, which must contain the origin as well.

The fact that a point set is in general position on $\mathbb{S}^{d-1}$ implies that any d points of A are contained in some *open* hemisphere $H \subset \mathbb{S}^{d-1}$. For $0 \leq k \leq d$, it makes sense to speak of a k-simplex of A, that is, the *spherical convex hull* of some $k+1$ points of A, which is denoted by $\text{conv}_{\mathbb{S}}$. More generally, if X is contained in some open hemisphere, then $\text{conv}_{\mathbb{S}} X$ is the intersection of all open hemispheres that contain X.

Remark. This is not the standard definition of the spherical convex hull, and it might be more appropriate to call it the *strong* spherical convex hull. A subset $K \subset \mathbb{S}^{d-1}$ is called *strongly convex* if it is contained in an open hemisphere, and for every pair of points x and y in K, the shortest arc from x to y is contained in K (see [6]). By this definition, we require a set to be contained in some open hemisphere in order to have a convex hull. It is also important to note that a point set in general position on $\mathbb{S}^{d-1}$ contains the origin in its *affine convex hull* if and only if it is not contained in any open hemisphere. Thus the point sets that contain the origin in their affine convex hull are *precisely* the ones for which the spherical convex hull is undefined.

Lemma 9. *For $d \geq 2$, let $A_1, \ldots, A_d$ be disjoint, non-empty, finite point sets on $\mathbb{S}^{d-1}$ such that $A_1 \cup \cdots \cup A_d$ is in general position, and suppose $A_i \cup A_j$ is not contained in any open hemisphere, for any $1 \leq i < j \leq d$. Let U denote the union of all $(d-1)$-simplices spanned by the d-tuples consisting of a single point from each A_i. If U does not cover $\mathbb{S}^{d-1}$, then for some $1 \leq i \leq d$ there exists an open hemisphere H such that $A_i \subset H \subset U$.*

The proof of Lemma 9 requires a basic topological fact (Claim 10, below) which follows by considering the (Brouwer) degree of a mapping $f : \mathbb{S}^k \to \mathbb{S}^k$, but for completeness we also give a direct proof.

For $d \geq 2$, let K be a finite collection of $(d-1)$-simplices on $\mathbb{S}^{d-1}$. A point $p \in \mathbb{S}^{d-1}$ will be called *generic with respect to K* provided that p is not contained in any of the faces of the simplices of K of dimension less than $d-1$. In other words, p is generic with respect to K if and only if p does not belong to the boundary of any member of K. For a generic point p, let the *order of p with respect to K* denote the number of $(d-1)$-simplices of K which contain p in their relative interiors. (We may omit the 'with respect to K' when it is clear from the situation what K is).

Claim 10. *For $k \geq 2$, let $B = \{a_1, \ldots, a_k, b_1, \ldots, b_k\}$ be distinct points in general position on $\mathbb{S}^{k-1}$. Let K denote the collection of all $(k-1)$-simplices formed by k-tuples of B with no repeated indices. Either the order of every generic point is even, or the order of every generic point is odd.*

Proof. Let L be the union of faces of the simplices of K, of dimension less than $k-1$. Then $\mathbb{S}^{k-1} \setminus L$ is a collection of finitely many nice open parts, and any pair of generic points can be connected by a path on $\mathbb{S}^{k-1}$ that does not pass through any faces of dimension less than $k-2$. Thus it suffices to consider how the order changes as we pass through a face of dimension $k-2$. For any face F of dimension $k-2$, there are precisely two points a_i and b_i (for a particular $1 \leq i \leq k$) such that $\text{conv}_{\mathbb{S}}(F \cup \{a_i\})$ and $\text{conv}_{\mathbb{S}}(F \cup \{b_i\})$ are $(k-1)$-simplices of K. Let H be the unique great $(k-2)$-sphere that contains F. If a_i and b_i are contained in the same open hemisphere bounded by H, then the order changes by ± 2 as we pass through F. If a_i and b_i are contained in opposite open hemispheres bounded by H, then the order stays the same as we pass through F. $\square$

Proof of Lemma 9. Suppose U does not cover $\mathbb{S}^{d-1}$. Since U is the union of finitely many simplices, U is closed and has a boundary which is a subset of finitely many $(d-2)$-faces of simplices of U. Let p be a point of the boundary of U with the property that it is contained in the relative interior of a unique $(d-2)$-face. Clearly such a point must exist, so suppose p is contained in the relative interior of the unique $(d-2)$-face, $F = \mathrm{conv}_{\mathrm{S}}\ \{a_1, \ldots, a_{d-1}\}$, where $a_i \in A_i$.

There is a unique great $(d-2)$-sphere, H, which contains the points $a_1, \ldots, a_{d-1}$ and bounds disjoint open hemispheres H^+ and H^-. If there exists points $x^+ \in A_d \cap H^+$ and $x^- \in A_d \cap H^-$, then p belongs to the $(d-1)$-simplices $\mathrm{conv}_{\mathrm{S}}\,(\{x^+\} \cup F)$ and $\mathrm{conv}_{\mathrm{S}}\,(\{x^-\} \cup F)$, which have disjoint relative interiors, share the common face F, and belong to U. This is impossible since p is a boundary point of U, so we may assume that $A_d \subset H^-$.

For every $1 \leq i \leq d-1$, we must have $A_i \cap H^+ \neq \emptyset$. If not, there exists an A_i such that $A_i \cup A_d \subset H \cup H^-$, which, by the general position assumption, means that $A_i \cup A_d$ is contained in some open hemisphere. Pick points $a \in A_d \subset H^-$ and $p_i \in A_i \cap H^+$, and let $b_i = H \cap \mathrm{conv}_{\mathrm{S}}\,\{p_i, a\}$. It follows from the general position assumption that the set of points $J = \{a_1, \ldots, a_{d-1}, b_1, \ldots, b_{d-1}\}$ is in general position on H. Let K denote the set of $(d-2)$-simplices spanned by the $(d-1)$-tuples of J with no repeated indices. By our choice of p, it follows that p is a generic point in H with respect to K.

Let $G \neq F$ be a $(d-2)$-simplex of K. It follows from how we defined the points of J, that any point in the relative interior of G is contained in the relative interior of a $(d-1)$-simplex spanned by U, for instance,

$$x \in \mathrm{int}\,\mathrm{conv}_{\mathrm{S}}\,\{a_1, a_2, b_3, b_4, \ldots, b_{d-1}\} \subset \mathrm{int}\,\mathrm{conv}_{\mathrm{S}}\,\{a, a_1, a_2, p_3, p_4, \ldots, p_{d-1}\}.$$

This means that p is covered only once (in H) by the $(d-2)$-simplices of K, and hence has order 1. So by Claim 10, with $k = d-1$, the simplices of K must cover H, and therefore

$$A_d \subset (H^- \cup H) \subset \bigcup_{X \in K} \mathrm{conv}_{\mathrm{S}}\,(\{a\} \cup X),$$

which completes the proof. $\square$

Proof of Theorem 8. The sets $A_1, \ldots, A_d$ are in general position on $\mathbb{S}^{d-1}$ and satisfy the conditions of Lemma 9. Thus they define the set U. If there exists a point $a \in A_{d+1}$ such that $\{-a\} \cap U \neq \emptyset$, then $-a$ is contained in some $(d-1)$-simplex, S of U, which means that the origin is contained in $\mathrm{conv}\,(\{a\} \cup S)$. On the other hand, if $(-A_{d+1}) \cap U = \emptyset$, then U cannot cover $\mathbb{S}^{d-1}$, so by Lemma 9 there is some $1 \leq i \leq d$ and an open hemisphere H such that $A_i \cup A_{d+1} \subset H \subset U$, which is a contradiction. $\square$

5 A Colored Spherical Helly Theorem.

An open hemisphere $H \subset \mathbb{S}^{d-1}$ determines an antipodal pair of normal vectors, and the one contained in H we refer to as the *pole* of H.

The Spherical Helly Theorem states that for any collection $\mathcal{C}$ of compact strongly convex sets on $\mathbb{S}^{d-1}$, there is a point in common to every member of $\mathcal{C}$ if and only if every $d+1$ or fewer members of $\mathcal{C}$ have a point in common. This can be proved as a consequence of Helly's theorem in $\mathbb{R}^d$. Moreover, if the members of $\mathcal{C}$ have no point in common, there exist open hemispheres $H_1, \ldots, H_{d+1}$ and sets $K_1, \ldots, K_{d+1}$ in $\mathcal{C}$ such that $K_i \subset H_i$ and $H_1 \cap \cdots \cap H_{d+1} = \emptyset$. This in turn implies that the poles of the H_i are not contained in any open hemisphere.

The Colored Helly Theorem states that if $\mathcal{C}_1, \ldots, \mathcal{C}_{d+1}$ are collections of compact convex sets in $\mathbb{R}^d$, and every selection of sets from distinct $\mathcal{C}_i$ have a point in common, then, for some $1 \leq i \leq d+1$, there is a point in common to every member of $\mathcal{C}_i$.

We can now state the dual version of Theorem 5. It can be interpreted as a colored version of the Spherical Helly Theorem, and it implies the Colored Helly Theorem.

Theorem 11. *Let $\mathcal{C}_1, \ldots, \mathcal{C}_{d+1}$ be non-empty, finite collections of compact strongly convex sets on $\mathbb{S}^{d-1}$. If every selection of sets from distinct $\mathcal{C}_i$ have a point in common, then for some $1 \leq i < j \leq d+1$ there is a point in common to every member of $\mathcal{C}_i \cup \mathcal{C}_j$.*

Proof. Suppose for contradiction that for each pair of collections $\mathcal{C}_i, \mathcal{C}_j$ ($1 \leq i < j \leq d+1$), the intersection of the members of $\mathcal{C}_i \cup \mathcal{C}_j$ is empty. By the Spherical Helly Theorem there exist sets $K_1, \ldots, K_{d+1} \in \mathcal{C}_i \cup \mathcal{C}_j$, and open hemispheres $H_1, \ldots, H_{d+1}$ with $K_i \subset H_i$ such that $H_1 \cap \cdots \cap H_{d+1} = \emptyset$. We associate the open hemispheres with points $a_1, \ldots, a_{d+1}$ (the poles of the H_i), and it follows that they are not contained in any open hemisphere. We define sets A_i and A_j as follows: Let the point $a_m \in A_i$ if $K_m \in \mathcal{C}_i$ and $a_m \in A_j$ if $K_m \in \mathcal{C}_j$. This gives us point sets $A_1, \ldots, A_{d+1}$, at most one of which is empty. If there is an A_i that is empty, choose any $K \in \mathcal{C}_i$ and an open hemisphere H such that $K \subset H$. Let a be the pole of H and set $A_i = \{a\}$.

By construction, the sets A_i are non-empty and they satisfy the condition that $A_i \cup A_j$ is not contained in any open hemisphere. By Theorem 5, there exist points $a_1 \in A_1$, $\ldots$, $a_{d+1} \in A_{d+1}$ that are not contained in any open hemisphere. This implies that the corresponding hemispheres $H_1, \ldots, H_{d+1}$ have no point in common, which is a contradiction, since each H_i contains a $K \in \mathcal{C}_i$. □

We show that Theorem 11 implies the Colored Helly Theorem. Given collections $\mathcal{C}_1, \ldots, \mathcal{C}_{d+1}$, let D be a large disk that contains every member of $\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_{d+1}$. If there is a point in common to every member of a subcollection of $\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_{d+1}$, then this point must be in D as well. Let $\mathcal{C}_{d+2} = \{D\}$.

We regard $\mathbb{R}^d$ as an affine hyperplane in $\mathbb{R}^{d+1}$ (which does not pass through the origin) and centrally project the members of $\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_{d+2}$ onto $\mathbb{S}^d$. This provides a bijection between $\mathbb{R}^d$ and the points of an open hemisphere of $\mathbb{S}^d$. If the original collections satisfy the hypothesis of the Colored Helly Theorem, then the projected collections will satisfy the hypothesis of Theorem 11, and since we can find a point in common to every member of *two* collections, one of these collections must be different from $\mathcal{C}_{d+2}$. So indeed Theorem 11 implies Lovász' colored Helly theorem.

Remark. The previous argument implicitly shows that Theorem 11 (and thus Theorem 5, too) is tight in the sense that we cannot hope for a point in common to *three* of the collections. If such a statement were true, then it would imply that we could find a point in common to *two* of the collections in Lovász' colored Helly theorem, but this is clearly not true. It is also easy to see that if we reduce the number of collections in Theorem 11, then there may not be a point in common to every member of even a *single* collection.

Remark. While preparing this manuscript we were made aware that Theorems 5 and 11 were independently discovered by J. L. Arocha, I. Bárány, J. Bracho, R. Fabila and L. Montejano.

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