The number of distinct distances from a vertex of a convex polygon

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Abstract

Erdős conjectured in 1946 that every n-point set P in convex position in the plane has a point that determines at least $\lfloor n/2 \rfloor$ distinct distances to the other points of P. In 2006 Dumitrescu improved the best known lower bound for this problem, from $n/3$ to $13n/36 - O(1)$. A crucial step in his argument is showing that P must determine at most $n^2(1 - 1/12)$ isosceles triangles.

In this paper we show that Dumitrescu's bound can be further improved, though our improvement is quite small. We show that the number of isosceles triangles determined by P is at most $n^2(1 - 1/11.981)$, and we conclude that there exists a point of P that determines at least $(\frac{13}{36} + \frac{1}{22701} - o(1))n$ distinct distances.

1 Introduction

We say that a point set P *determines* a distance d if P has two elements such that their Euclidean distance is d. Given a positive integer n, what is the maximum number $g(n)$ such that every set of n points in the plane determines at least $g(n)$ distinct distances? According to a famous conjecture of Erdős [E46], we have $g(n) = \Omega(\frac{n}{\sqrt{\log n}})$. The number of distinct distances determined by a $\sqrt{n} \times \sqrt{n}$ piece of the integer lattice is $O(\frac{n}{\sqrt{\log n}})$, which shows that his conjecture, if true, would be essentially best possible.

In a recent breakthrough, Guth and Katz [GK11] have come very close to proving Erdős's conjecture. They showed that $g(n) = \Omega(\frac{n}{\log n})$. This is a substantial improvement on the previous bound of $g(n) \geq n^{0.864\ldots}$ by Katz and Tardos [KaT04], which was the last step in a long series of successive results [Mo52], [Ch84], [ChST92], [Sz93], [SoT01], [Ta03].

In the same paper, Erdős [E46] also made a much stronger conjecture. Let $f(n)$ denote the maximum number such that every set of n points in the plane has a point from which there are at least $f(n)$ distinct distances to the other $n-1$ points of the set. Clearly, we have $f(n) \leq g(n)$. Erdős conjectured that in fact $f(n) = \Omega(\frac{n}{\sqrt{\log n}})$. This conjecture is still wide open, although all the

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above mentioned lower bounds, with the exception of the one due to Guth and Katz, also apply to $f(n)$. In particular, the best known lower bound of $f(n)$ is still $f(n) \geq n^{0.864\dots}$ by Katz and Tardos.

As Erdős suggested, the same question can be studied for point sets with special properties. We say that n points in the plane are in *convex position* if they form the vertex set of a convex n-gon. A set of n points is in *general position* if no 3 of its elements are collinear. Let $f_{\text{conv}}(n)$ (and $f_{\text{gen}}(n)$) denote the largest number such that every set of n points in the plane in convex (resp., in general) position in the plane has a point from which there are at least these many distinct distances to the remaining $n-1$ points. Since every set in convex position is also in general position, we have $f_{\text{gen}}(n) \leq f_{\text{conv}}(n)$. The vertex set of a regular n-gon shows that

$$f_{\text{gen}}(n) \leq f_{\text{conv}}(n) \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

Erdős conjectured that $f_{\text{conv}}(n) = \lfloor \frac{n}{2} \rfloor$ for all $n \geq 2$. It is perfectly possible that the same equality holds for $f_{\text{gen}}(n)$. The weaker statement that every set of n points in convex position determines $\lfloor \frac{n}{2} \rfloor$ distinct distances was proved by Altman [Al63], [Al72]. Leo Moser [Mo52] proved that $f_{\text{conv}}(n) \geq \frac{n}{3}$, while Szemerédi established essentially the same lower bound for point sets in general position: By a very simple double-counting argument, he established the inequality $f_{\text{conv}}(n) \geq f_{\text{gen}}(n) \geq \frac{n-1}{3}$; see [E75], [PaA95].

By combining and improving the arguments of Moser and Szemerédi, Dumitrescu [Du06] established the bound

$$f_{\text{conv}}(n) \geq \left\lceil \frac{13n-6}{36} \right\rceil.$$

In the present note, we show that Dumitrescu's bound can be further improved.

Theorem 1. *The minimum number $f_{\text{conv}}(n)$ of distinct distances determined by n points in convex position in the plane satisfies*

$$f_{\text{conv}}(n) \geq \left(\frac{13}{36} + \varepsilon - o(1) \right) n,$$

for a suitable positive constant ε.

Our argument as presented here yields a little over $\epsilon > 1/23000$. It is quite possible that this bound can be slightly improved through tweaks in different places, though we have abstained from doing so in the interest of simplicity.

As we shall see later, the crucial point in the argument of Szemerédi is estimate in two different ways the number $Z(P)$ of isosceles triangles determined by a set P of points. In case P is a set of points in general position an upper bound of $2\binom{n}{2}$ for $Z(P)$ follows easily. The argument of Dumitrescu in [Du06] established an upper bound of $n^2(1-1/12)$ for $Z(P)$, in the case where P is a set of n points in *convex position* in the plane, thus enabling him to improve the lower bound for $f_{\text{conv}}(n)$.

In this paper we further improve this upper bound of Dumitrescu for the number of isosceles triangles determined by n points in convex position in the plane. Since this is an independently interesting problem, we sate it explicitly:

Problem 2. What is the minimal number $I_{\text{conv}}(n)$ ($I_{\text{gen}}(n)$) such that any set of n points in convex (general) position in the plane determines at most $I_{\text{conv}}(n)$ ($I_{\text{gen}}(n)$) isosceles triangles? (Here we count every equilateral triangle three times.)

To make our paper self-contained, in the next section we briefly sketch and later use the arguments of Moser, Szemerédi, and Dumitrescu. In Section 3, we prove three auxiliary results on the number of isosceles triangles induced by a point set in convex position. At a crucial point of the proof, we need to answer a variant of the following question of independent interest, concerning arithmetic progressions of length 3:

Problem 3. Let $x_1 < x_2 < \ldots < x_n < y_1 < y_2 < \ldots < y_n$ be a sequence of reals. A pair (x_i, y_j) is called "bad" if $\frac{x_i+y_j}{2}$ is equal to x_k or y_k for some k ($1 \leq k \leq n$). What is the maximum number of bad pairs (x_i, y_j) over all sequences of length $2n$?

In Section 4, we will give a partial answer to a generalization of this question and show how it yields a modest improvement of Dumitrescu's result. Section 5 contains some open problems and concluding remarks.

2 The arguments of Szemerédi, Moser, and Dumitrescu

First, we sketch Szemerédi's argument to prove the inequality $f_{\text{gen}}(n) \geq \frac{n-1}{3}$. Let P be a set of n points in general position in the plane, and assume that for every element of $p \in P$, the number of distinct distances to the other $n-1$ points is at most k. Let $Z(P)$ denote the number of isosceles triangles induced by the elements of P, that is, the number of unordered pairs $\{(p,a),(p,b)\}$ such that $p, a, b \in P$ and $|pa| = |pb|$. (Here $|pa|$ stands for the length of segment pa.) Note that, according to this convention, an equilateral triangle abc is counted *three* times as an isosceles triangle: When the role of p is played by a, by b, or by c.

Clearly, we have $Z(P) \leq 2\binom{n}{2}$, because for each pair a, b there exist at most two points $p \in P$ with $|pa| = |pb|$. This follows from the fact that all such points p must lie on the perpendicular bisector of ab, and P has no three collinear points. On the other hand, using the convexity of the function $\binom{x}{2}$, for every point $p \in P$, the number of pairs $\{a, b\}$ with $|pa| = |pb|$ is minimized when the $n-1$ points of $P \setminus \{p\}$ are distributed among the at most k concentric circles around p as equally as possible. That is, the number of such pairs $\{a, b\}$ is at least $k\binom{\frac{n-1}{k}}{2}$. Comparing the two bounds, we obtain

$$nk\binom{\frac{n-1}{k}}{2} \leq Z(P) \leq 2\binom{n}{2},$$

which yields that $k \geq \frac{n-1}{3}$. Hence, we have

$$f_{\text{gen}}(n) \geq \frac{n-1}{3},$$

as stated.

It is obvious from the above argument that if we manage to improve the upper bound on $Z(P)$, then we obtain a better lower bound on the largest number of distinct distances measured from a point of P.

Lemma 4. *Suppose that the number $Z(P)$ of isosceles triangles determined by an n-element point set P in the plane satisfies $Z(P) \leq \alpha n^2 + O(n)$ for some $\alpha \leq 1$. Then P has a point from which there are at least $\frac{2-\alpha}{3}n - O(1)$ distinct distances.*

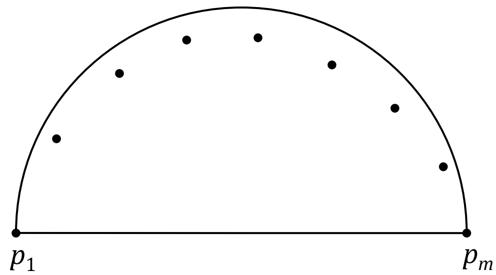

Figure 1: A cap.

Proof. Assume, as above, that for every point $p \in P$, the remaining $n-1$ points lie on at most k circles centered at p. By Szemerédi's proof, we also know that $2 \leq \frac{n-1}{k} \leq 3$. Otherwise, we have $k \geq n/2$, and we are done.

This means that for each point $p \in P$, the number of pairs $\{a, b\}$ with $|pa| = |pb|$ is minimized when there are precisely k circles around p that pass through at least one element of P, and each of these circles contains either 2 or 3 points. Since $n-1 = (3k-n+1)2+(n-1-2k)3$, we can assume that in the worst case $3k-n+1$ circles contain 2 points and $n-1-2k$ circles contain 3 points. Thus, the number of pairs $\{a, b\}$ with $|pa| = |pb|$ is at least $(3k-n+1)+(n-1-2k)3 = 2(n-1)-3k$. Therefore, $Z(P) \geq n(2(n-1)-3k)$. Combining this with the upper bound on $Z(P) \leq \alpha n^2 + O(n)$, the lemma follows. $\square$

Remark. Taking $n-1$ points $x_1, \ldots, x_{n-1}$ evenly distributed on say quarter of a circle together with the center of the circle x_n, we get an example of a set P of n points in convex position such that $Z(P) = 3n^2/4 + O(n)$. Hence the method described in Lemma 4 can yield at best a lower bound of $5n/12 + O(1)$ for $f_{\text{conv}}(n)$, rather than the conjectured bound. This construction shows also that $I_{\text{gen}}(n) \geq I_{\text{conv}}(n) \geq 3n^2/4 + O(n)$ in Problem 2.

Dumitrescu [Du06] showed that if P is a set of n points in *convex* position, then $Z(P) \leq \frac{11}{12}n^2$. Plugging this bound into Lemma 4, we obtain that $f_{\text{conv}}(n) \geq \frac{13}{36}n - O(1)$.

In the present note, we slightly improve on Dumitrescu's upper bound on $Z(P)$ for point sets in convex position, and hence on his lower bound for $f_{\text{conv}}(n)$. For this, we first recall the terminology of Moser [Mo52] and Dumitrescu [Du06].

Definition 5. A set of points Q in convex position is called a *cap with endpoints a and b* if the elements of Q can be enumerated in cyclic order, as $x_1, x_2, \ldots, x_t$, such that $x_1 = a, x_t = b$ and there is a circle C passing through a and b such that all x_i lie in the closed region bounded by ab and the shorter arc of C delimited by a and b. (If the two arcs of C are of the same length, either of them will do. See Figure 1.)

It is not hard to see that $x_1, x_2, \ldots, x_t$ form a cap if and only if $\measuredangle x_1 x_i x_t \geq \frac{\pi}{2}$ for all $1 < i < t$. Using the convexity of the set of the point set, this is further equivalent to the condition that $\measuredangle x_i x_j x_k \geq \frac{\pi}{2}$ for every $1 \leq i < j < k \leq t$. This implies:

1. For every cap $x_1, x_2, \ldots, x_t$, we have
$$|x_1x_2| < |x_1x_3| < \ldots < |x_1x_t|.$$
(Indeed, since $\measuredangle x_1 x_i x_{i+1}$ is the largest angle in the triangle $x_1 x_i x_{i+1}$, we have $|x_1 x_{i+1}| > |x_1 x_i|$.)

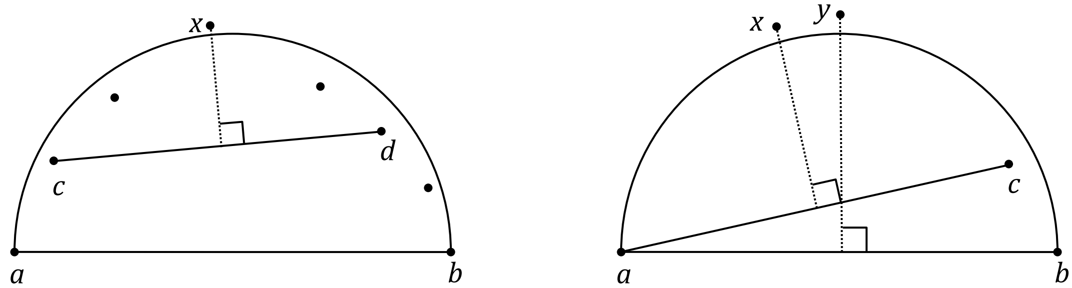

Figure 2: Left: A witness for an edge in a cap (note that the witness does not necessarily belong to the cap). Right: Witnesses for two edges in a cap sharing a common vertex.

2. Every subsequence of a cap is also a cap.

Moser [Mo52] noticed that the smallest circumscribing circle around a set P in convex position divides it into at most three caps that meet only at their endpoints. At least one of them has length $t \geq \lceil \frac{n}{3} \rceil + 1$. Therefore, property 1 above yields that $f_{\text{conv}}(n) \geq \lceil \frac{n}{3} \rceil$.

Definition 6. Let P be a set of points in convex position. An unordered pair (edge) $\{a, b\} \subset P$ is called *good* if the perpendicular bisector of the segment ab passes through at most one point of P. Otherwise, it is called *bad*. The pair (edge) $\{a, b\}$ will be often identified with the segment $ab = ba$.

Definition 7. Let $Q \subset P$ be a cap with endpoints a and b, and let $c, d \in Q$. A point $x \in P$ is called a *witness for the edge* cd if x lies on the perpendicular bisector of the segment cd, and the line ab does not separate x from the points of Q.

Since P is in convex position, the witness x for an edge cd, if exists, is uniquely determined. The witness x for cd must lie between the two points c and d, in the sense that the line cd must separate x from the points a and b (see Figure 2, left).

The following lemma is a stronger version of a statement from [Du06].

Lemma 8. *Let $Q \subset P$ be a cap with endpoints a and b. Let c be a point of Q and assume that x and y from P are witnesses for ac and ab, respectively. Then x lies between a and y, in the sense that the line through a and y separates x from b. In particular, we have $x \neq y$. See Figure 2, right.*

Proof. Assume without loss of generality that ab is horizontal, a is to the left of b, and Q lies above the line ab. Assume to the contrary that y lies between a and x, or $y = x$. We know already that x lies between a and c. We have $|yc| \geq |ya| = |yb|$. Therefore, we have $\measuredangle ycb < \pi/2$. However, we know that $\measuredangle acb < \measuredangle ycb$, contradicting the fact that $\measuredangle acb \geq \pi/2$, as Q is a cap. □

Janos says: It is somewhat strange that the cap in the above lemmas is denoted by Q, but in the next statements by P. ⟵

Gabriel says: The best thing is for P to always be the given point set and for Q to always be a cap. I changed it this way everywhere. ⟵

Corollary 9. *Let Q be a cap that consists of t points. Then there are at most $\frac{1}{4}t^2$ edges in Q that have a witness in Q.*

Proof. Denote the points of Q in cyclic order by $x_1, x_2, \ldots, x_t$. By Lemma 8, no two edges of Q that share a common vertex can have the same witness. Therefore, x_i can witness at most $\min(i-1, n-i)$ edges in Q. Hence, the number of edges in Q with a witness in Q is at most $2(1+2+\ldots+\lceil \frac{t}{2}-1 \rceil) \leq \frac{1}{4}t^2$. □

Corollary 10. [Du06] *Let P be a set of n points in convex position in the plane. Then P has at least $\frac{n^2}{12}$ good edges.*

Proof. The smallest enclosing circle C of P passes through (at most) 3 points $a, b, c \in P$ (possibly not all distinct) such that each of the arcs delimited by them is at most a semi-circle. Thus, a, b, and c divide P into at most 3 caps.

If a, b, c are distinct, let r, s, and t denote the number of points in these caps, where $r+s+t = n+3$. By Corollary 9, the total number of good edges completely contained in one of the caps is at least

$$\frac{1}{4}(r^2+s^2+t^2) \geq \frac{1}{4}3\frac{n^2}{9} = \frac{n^2}{12}.$$

If $b = c$, say, we obtain an even better lower bound. □

In order to complete Dumitrescu's argument, notice that if xy is a good edge in one of the 3 caps, then there is at most one isosceles triangle with base xy. Thus, we have

$$Z(P) \leq 2\binom{n}{2} - \#\{\text{good edges}\}.$$

According to Corollary 10, this implies $Z(P) < \frac{11}{12}n^2$. Plugging this bound into Lemma 4, we obtain that P determines at least $\frac{13}{36}n - O(1)$ distinct distances, which is Dumitrescu's theorem [Du06].

3 Three lemmas on witnesses

To improve the lower bound $f_{\text{conv}}(n) \geq \frac{13}{36}n$, we need a couple of auxiliary results. The first such statement is a simple consequence of Lemma 8.

Lemma 11. *Let Q be a cap of size t with endpoints a and b. Then the total number of edges adjacent to a and b with no witness in Q is at least $t-1$.*

Proof. Let x be the witness in Q for the edge ab, if it exists; otherwise add such a point x keeping $Q \cup \{x\}$ in convex position.

By Lemma 8, all witnesses to edges adjacent to b are between b and x, while all witnesses for edges adjacent to a are between a and x. We know already that a point in Q can be a witness for at most one edge adjacent to a and at most one edge adjacent to b. We conclude that every point in $Q \setminus \{a, b\}$ may be a witness for at most one edge adjacent to a or to b. As there are $2t-3$ edges in total that are adjacent to a or to b, there must be at least $2t-3-(t-2) = t-1$ edges in Q with no witness in Q. □

The following geometric lemma, which is of independent interest, will be a crucial element of our proof.

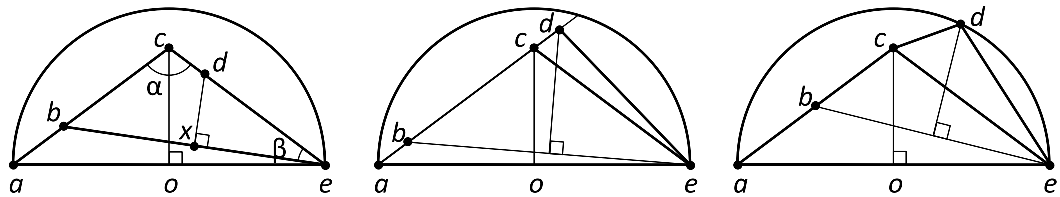

Figure 3: The three cases for Lemma 12.

Lemma 12. *Let $Q = \{a, b, c, d, e\}$ be a cap, with the points appearing in that order, such that c is a witness for ae and d is a witness for be. Then $|ab| > |cd|$.*

Proof. First, we can assume without loss of generality that b lies on the segment ac;[1] for otherwise, we can slide b counterclockwise along the circle centered at d passing through b, until b reaches ac, and this only decreases $|ab|$ (in fact, $|ab|$ keeps decreasing until b reaches the segment ad).

Next, let o be the midpoint of ae, let C be the circle centered at o passing through a and e, and let ℓ be the line passing through d perpendicular to be. Without loss of generality we can slide d along ℓ either inwards or outwards, making sure Q is still a cap, so as to maximize $|cd|$. Then, d falls in one of these three cases (see Figure 3):

1. d lies on ce.
2. d lies on the line through a and c.
3. d lies on C.

Suppose the first case. Let $x = \ell \cap be$, and let $\alpha = \measuredangle ace$ and $\beta = \measuredangle bec$. Then, by applying the law of sines on the triangle bce and considering the right-angled triangle xde, we get $|bc|/\sin\beta = |be|/\sin\alpha = 2|de|\cos\beta/\sin\alpha$, so $|bc| = |de|\sin(2\beta)/\sin\alpha = |de|\sin(2\beta)/\sin(\pi-\alpha)$. But $\pi - \alpha = 2\measuredangle cea > 2\beta$, which implies that $|bc| < |de|$, or equivalently $|ab| > |cd|$.

Now suppose the second case. Then $|ab| > |cd|$ is equivalent to $|ac| > |bd|$. But $|ac| = |ce|$ and $|bd| = |de|$. Furthermore, $|ce| > |de|$ since $\measuredangle cde \ge \pi/2$ in the triangle cde, so we are done.

The third case is divided into two subcases, according to whether d lies higher or lower than c (meaning, whether $\measuredangle dco$ is obtuse or acute).

If d lies higher than c, then without loss of generality we can move c down towards o, and move b counterclockwise along the circle centered at d, until both c and b reach the segment ad. This only decreases $|ab|$ and increases $|cd|$, and we fall back into case 2.

Finally, suppose that d lies on C but lower than c. Let b' be the point along ad satisfying $|b'd| = |bd|$; and let c' be the intersection point of C and the ray $\overrightarrow{oc}$. Note that $|ab| \ge |ab'|$ and $|cd| \le |c'd|$. We show algebraically that $|ab'| > |c'd|$, which proves our claim:

Suppose without loss of generality that o is the origin and C has radius 1. Let $d = (x, \sqrt{1-x^2})$ for some $0 < x < 1$. Then $|ab'| = |ad| - |b'd| = \sqrt{2+2x} - \sqrt{2-2x}$, while $|c'd| = \sqrt{2-2\sqrt{1-x^2}}$, and a routine algebraic calculation shows that $|ab'| > |c'd|$ for all $x > 0$. $\square$

[1] Then Q is only "weakly" convex, but this does not present any problem.

Lemma 13. *Let $Q = \{x_1, x_2, \dots, x_{2n}\}$ be a cap with the points appearing in that order. Then the number of edges between the sets $\{x_1, x_2, \dots, x_n\}$ and $\{x_{n+1}, x_{n+2}, \dots, x_{2n}\}$ that have a witness in Q is at most $\frac{7}{8}n^2 + O(n)$.*

Proof. Let G denote the "geometric graph" whose vertices are the points of Q and whose edges are those edges whose number we wish to bound in the lemma. Consider the set of segments $\{x_i x_{i+1} \mid i \neq n, 1 \leq i \leq 2n-1\}$, and let $s_1, s_2, \dots, s_{2n-2}$ denote these segments enumerated in increasing order of their lengths (i.e., we have $|s_1| \leq |s_2| \leq \cdots \leq |s_{2n-2}|$). For every $1 \leq i \leq 2n-2$, denote by u_i and v_i the endpoints of s_i so that $u_i = x_j$ and $v_i = x_{j+1}$ for some j.

We claim that $d(u_i) + d(v_i) \leq n + i$ for $i = 1, 2, \dots, n$, where $d(v)$ is the degree of vertex v in G.

Indeed, fix $1 \leq i \leq n$. Suppose without loss of generality that $v_i = x_j$ for some $j \leq n$. Let x_k be a vertex with $k > n$ such that both $u_i x_k$ and $v_i x_k$ are in G. Let their witnesses be x_ℓ and $x_{\ell'}$, respectively, with $\ell' > \ell$. Then, by Lemma 12, we have $|x_\ell x_{\ell+1}| \leq |x_\ell x_{\ell'}| < |s_i|$. Therefore, either $\ell = n$ or $x_\ell x_{\ell+1} = s_{i'}$ for some $i' < i$. Obviously, there are only $i - 1$ such segments $s_{i'}$. Furthermore, by Lemma 8, different edges $u_i x_k$, $u_i x_{k'}$ must have different witnesses. It follows that there can be at most i vertices among $x_{n+1}, \dots, x_{2n}$ that are connected to *both* u_i and v_i in G, so that $d(u_i) + d(v_i) \leq n + i$, as claimed.

Adding up over all segments, we obtain

$$\begin{aligned} 4|G| - 4n = 2\sum_{i=1}^{2n} d(x_i) - 4n &\leq \sum_{i=1}^{2n} d(x_i) - \big(d(x_1) + d(x_n) + d(x_{n+1}) + d(x_{2n})\big) \\ &= \sum_{i=1}^{2n-2} (d(u_i) + d(v_i)) \\ &\leq (n+1) + \cdots + 2n + \cdots + 2n = \frac{7}{2}n^2 - \frac{7}{2}n, \end{aligned}$$

and the lemma follows. □

We conjecture that the number of edges in G in Lemma 13 is in fact at most $n^2/2 + O(n)$. If true, this would yield a small improvement in Theorem 1.

Problem 3, mentioned in the Introduction, is a special limit case of Lemma 13, and even for it we do not have any bound better than $\frac{7}{8}n^2 + O(n)$.

4 Proof of the main result

Theorem 14. *Let P be a set of n points in convex position. Then P has at least αn^2 good edges, where $\alpha = 1/11.981$.*

Proof. Let $p_1, \dots, p_n$ be the points of P in circular order. In this proof, by the *circular distance* between two points $p_i, p_j \in P$ we mean the minimum of $(j - i) \bmod n$ and $(i - j) \bmod n$.

Fix two constants $0 < a, d < 1$ to be determined later. We will think of d as much smaller than a. Perform the following dn steps: Let $P_1 = P$. At step i choose the smallest enclosing circle of P_i and let x_i, y_i, z_i be the three points of P_i through which this circle passes. (If the circle passes through only two points, let $x_i = y_i$.) Then let $P_{i+1} = P_i \setminus \{x_i, y_i, z_i\}$.

We consider two cases:

Case 1. For some index i, $1 < i \le dn$, some point among x_i, y_i, and z_i is at circular distance at least an from each of the three points x_1, y_1, and z_1. Without loss of generality let x_i be that point.

GABRIEL SAYS: Rom, alert! I changed it around (switched the roles of 1 and i) becaue I think this is the correct way. Please check. ⟵

Note that P_i is partitioned into three caps in two different ways: The points x_1, y_1, z_1 define caps Q_1, Q_2, Q_3, while the points x_i, y_i, z_i define caps Q'_1, Q'_2, Q'_3.

The intuition for this case is that, since x_i is significantly far from x_1, y_1, z_1, these two partitions are, in a sense, significantly different.

Applying the argument of Corollary 10 to the caps Q'_1, Q'_2, Q'_3, we find at least $(n-3i)^2/12 \ge (n-3dn)^2/12$ edges that are good in P_i and connect points within the same cap. However, not all these edges are necessarily good in P, since the points in $P \setminus P_i$ might invalidate some of these edges.

However, by Lemma 8, no point of $P \setminus P_i$ can invalidate two adjacent edges in the same cap, so each point of $P \setminus P_i$ invalidates at most $n/2$ of these edges. Thus, we are left with at least

$$(n-3dn)^2/12 - 3dn^2/2$$

edges that are good in P and are *internal* to Q'_1, Q'_2, or Q'_3.

ROM SAYS: In the next paragraph there was an error due to the change Gabriel did. I fixed it but it may cost us a little in the constant. one has to do the optimization again. ⟵

SHIRA SAYS: I did the optimization again: the parameters have changed a bit but the result stays the same. ⟵

In addition, out of the $2an$ points of P at circular distance at most an from x_i there are at least $2an - 3dn$ points of P_i. All those points are contained in the same cap Q_1, Q_2, or Q_3. Applying Lemma 13 to them, we find at least another $(a-3d)^2n^2/8$ good edges in P which were not counted previously, since they straddle two different caps among Q'_1, Q'_2, Q'_3.

Hence, in case 1 we find at least

$$(n-dn)^2/12 + \frac{1}{8}(a-3d)^2n^2 - 3dn^2/2$$

good edges in P.

Case 2. For every index i between 1 and dn, each of x_i, y_i, z_i is at circular distance at most an from one of x_1, y_1, z_1 in P.

GABRIEL SAYS: As a bonus, we avoid the complication that was here. ⟵

In this case the analysis is somewhat different. It follows from Lemma 11 that at each step i the points x_i, y_i, and z_i together are adjacent to at least $n-3i$ good edges in P_i. As before, not all these edges are necessarily good in P. However, by Lemma 8, each vertex in $P \setminus P_i$ can invalidate at most one such edge. Hence, we are left with at least $n-6i$ good edges in P.

Now consider P_{dn}, and consider its partition into three caps Q_1, Q_2, Q_3 by the original three points x_1, y_1, z_1. By Corollary 10, there are at least $(n-3dn)^2/12$ edges that are good in P_{dn} and connect two points within the same cap Q_1, Q_2, or Q_3.

As before, not all these edges are necessarily good in P, but we can bound the number of edges invalidated by the points x_i, y_i, z_i, $i < dn$ of $P \setminus P_i$: Each such point is within circular distance an of x_1, y_1, or z_1, so by Lemma 8, it can invalidate at most an of these edges.

Therefore, in case 2 we find at least

$$\sum_{i=1}^{dn}(n-6i) + (n-3dn)^2/12 - 3dan^2 = n^2/12 + dn^2/2 - \frac{9}{4}d^2n^2 - 3dan^2 - O(n)$$

good edges in P.

If we choose properly a and d we can guarantee that in all cases we have strictly more than $n^2/12$ good edges. The values $a = 1/8.87$ and $d = 1/1204.5$ are close to the optimal ones, and they yield at least $n^2/11.981$ good edges. □

Plugging in $\alpha = 10.981/11.981$ into Lemma 4, we finally get

$$f_{\mathrm{conv}}(n) \geq \left(\frac{13}{36} + \frac{1}{22701} - o(1)\right) n,$$

proving Theorem 1.

5 Concluding remarks

GABRIEL SAYS: The special circles business? ⟵

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