Two and a Half Billion Years of Distance Research

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Abstract

In a seminal paper published in 1946, Erdős initiated the investigation of the distribution of distances generated by point sets in metric spaces. In spite of some spectacular partial successes and persistent attacks by generations of mathematicians, most problems raised in Erdős' paper are still unsolved. Given a set of n points in $\mathbb{R}^d$, let $d_1 > d_2 > d_3 > \cdots$ denote the sequence of all distances between pairs of points in P, listed in decreasing order. We raise some simple questions related to a famous conjecture of Schur. For instance, is it true that any two regular $(d-1)$-dimensional simplices of side length d_1 induced by P share at least one vertex? We prove that if P is the vertex set of a convex polygon in $\mathbb{R}^2$, then the maximum number of equilateral triangles of side length d_k induced by P is $\Theta(k)$.

1 Introduction

Many of Erdős' jokes were about "old age and stupidity." As a young man, he was already afraid of Alzheimer's disease and the decline of

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mental capacities. He was less than thirty when he started referring to himself as an “old man”. To mark the various stages of the process, he put more and more letters after his name: P.G.O.M. for Poor Great Old Man, and then L.D. for Living Dead, A.D. for Archeological Discovery, another L.D. for Legally Dead when he turned seventy, and C.D. for Counts Dead, five years later. (The last title refers to the former rule that the Hungarian Academy of Sciences could not have more than two hundred members, but people over seventy five did not count.) Erdős loved to argue that he was two and a half billion years old, because in his youth the age of the Earth was known to be two billion years and by the end of his life it was four and a half billion.

In this sense, the subject of this note is also two and a half billion years old. It started in 1934, in the year of the "Night of Long Knives," when Hitler was elected President of Germany. Hungary's prime minister at that time was Gyula Gömbös, the first head of a foreign country who paid an official visit to the Nazi leader. In the same year, *Heinz Hopf* and his student, *Erika Pannwitz*, at Friedrich Wilhelms University (today Humboldt University) in Berlin, posed the following problem in the problem section of *Jahresbericht der Deutschen Mathematiker-Vereinigung* [6]: Take a set P of n points in the plane and connect two of them by an edge if their distance is equal to the *diameter*, i.e., the largest distance determined by the point set. Prove that the resulting so called *diameter graph* $D(P)$ does not contain any cycle of even length. Correct solutions were submitted by W. Fenchel, R. Frucht, B. Neumann, L. Rédei, L. A. Santaló, and some other mathematicians, who became leading researchers after the war. Paul Erdős may have come across the September 1934 issue of the *Jahresbericht*, in which the above problem appeared. However, 1934 was a particularly busy year for him. He defended his thesis at Péter Pázmány University (today Loránd Eötvös University), Budapest. Because of the increasingly anti-semitic atmosphere in Hungary, in the same year, he accepted a fellowship arranged by *Louis J. Mordell*, and moved to Manchester.

Erdős was not only aware of the Hopf–Pannwitz problem, but he also made an important further observation: The solution easily implies that the maximum number of edges that the diameter graph of n points in the plane can have is n. In his classic 1946 paper [3] published in the *American Mathematical Monthly* he generously attributed this statement to Hopf and Pannwitz. Erdős' paper, alongside with another important note by him [1] that appeared one year earlier, were clearly inspired by the Hopf–Pannwitz problem. He extended the question to the investigation of other graphs defined on sets of points P in metric spaces (mostly Euclidean spaces), in which two points of P are connected by an edge if and only if their distance belongs to a fixed set Δ of special distances. (In the Hopf-Pannwitz problem, Δ is a one-element set consisting of the diameter of the point set.) The resulting new questions led to many exciting open problems in extremal graph theory, combinatorics, algebra, additive number theory, and other areas of mathematics. Although most problems of this type raised by Erdős are still open, there is little doubt that they have richly contributed to the development of these subjects.

2 Two generalizations of the Hopf–Pannwitz–Erdős result

Suppose that, instead of planar point sets, we consider n-element point sets P in higher-dimensional spaces. As Erdős reported in [3], Vázsonyi conjectured that if $P \subset \mathbb{R}^3$, $|P| = n > 3$, the diameter graph $D(P)$ of P has at most $2n-2$ edges. This was proved independently by Grünbaum [7], Heppes [8], and Straszewicz [14]. The bound $2n-2$ is best possible; see Figure 1. In dimensions larger than 3, the analogous problem turned out to have a different flavor: Lenz found some simple constructions with a quadratic number of diameters.

Schur suggested another possible extension of the Hopf–Pannwitz–Erdős result to higher dimensions (see [13]). Instead of estimating the number of edges, consider the number of cliques. A *k-clique*, that is, a complete subgraph of k vertices in the graph of diameters $D(P)$ generated by a set $P \subset \mathbb{R}^d$, corresponds to a regular $(k-1)$-dimensional simplex (or, in short, *$(k-1)$-simplex*) of side length $\mathrm{diam}(P)$. Schur formulated the following conjecture.

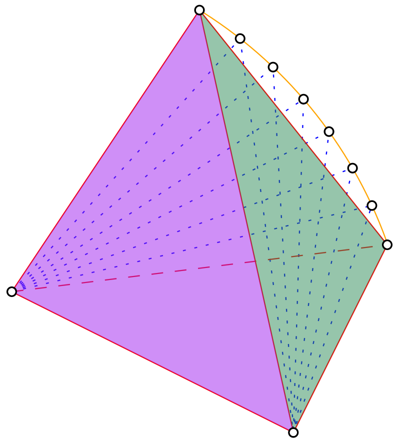

Figure 1: A construction with n points and $2n-2$ diameters in $\mathbb{R}^3$.

Conjecture 1 (Schur)**.** *Let P be a set of n points in $\mathbb{R}^d$, where $n > d > 1$. The number of d-cliques in the graph of diameters of P is at most n.*

This conjecture, if true, is best possible. This can be shown by a straightforward generalization of Vázsonyi's example depicted in Figure 1. Schur, Perles, Martini, and Kupitz [13] proved that Conjecture 1 is true in 3 dimensions. In other words, any set of $n > 3$ points in $\mathbb{R}^3$ can generate at most n equilateral triangles of side length $\mathrm{diam}(P)$.

In [10], we proved Schur's conjecture in any dimension, under the extra condition that any two d-cliques of the diameter graph $D(P)$ share at least $d-2$ vertices. We could not exclude the possibility that this condition is satisfied for every point set P. It follows from a theorem of Dolnikov [2] that this is the case when $d = 3$. Moreover, Dolnikov showed that a graph of diameters in $\mathbb{R}^3$ contains no two vertex disjoint odd cycles. To settle Conjecture 1 for $d \geq 4$, we need to answer the following simple question.

Question 2. *Let $d > 3$, and let S and S' be two regular $(d-1)$-simplices of side length 1 in $\mathbb{R}^d$, with the property that the diameter of $S \cup S'$ is 1. Is it true that S and S' must share at least $d-2$ vertices?*

In fact, we cannot even decide whether two such simplices must have at least *one* vertex in common.

Let us summarize what we know about diameter graphs $D(P)$ of finite point sets $P \subset \mathbb{R}^3$.

1. For any $P' \subseteq P$ with $|P'| = n' \geq 3$, the number of edges of the subgraph of $D(P)$ induced by P' is at most $2n' - 2$ (Vázsonyi).
2. Any two odd cycles of $D(P)$ must share a vertex (Dolnikov).
3. $D(P)$ contains no $K_{3,3}$ as a subgraph (Erdős [4]).
4. $D(P)$ contains at most one copy of K_4 (Schur et al. [13]).

It would be interesting to come up with an example of a graph that satisfies all these properties, but is not a graph of diameters. In particular, the following question can be asked.

Question 3. *Are all bipartite planar graphs diameter graphs in* $\mathbb{R}^3$*?*

3 Near-diameters

Instead of estimating the number of times the diameter occurs in a set of points, we may consider the number of occurrences of other *large distances*. Given a set P of n points in $\mathbb{R}^d$, let $d_1 > d_2 > d_3 > \cdots$ denote the sequence of all distances between pairs of points in P, listed in decreasing order. According to the Hopf–Pannwitz–Erdős result, the distance d_1 can occur at most n times. In 1987, Vesztergombi [16] showed that the second-largest distance, d_2, can occur at most $\frac{3}{2}n$ times. In another paper [17], she considered the version of the problem when the points of P are in *convex position*, and she proved that in this case the number of second-largest distances cannot exceed $\frac{4}{3}n$. Both of these bounds are tight up to additive constants. Some further bounds on the number of large distances (third and fourth largest distances) in convex polygons were obtained in [11]. In general, it is known that the number of k-th largest distances in the plane is at most $2kn$ [16]. A slightly better bound, kn, is known for point sets in convex position [17]. It is not clear how tight these bounds are for large k.

Less is known about graphs of large distances in 3 dimensions. We proved in [9] that, for a fixed k, the number of k-th largest distances among n points in $\mathbb{R}^3$ is still linear in n, i.e., at most $c_k n$ for a constant c_k.

In the spirit of Schur's generalization of the Hopf–Pannwitz–Erdős theorem, we may estimate the number of simplices with the property that all of their sides are "large." Schur et al. [13] proved that the graph of diameters contains at most one such full-dimensional simplex, for any finite set of points in $\mathbb{R}^d$ (cp. property 4 diameter graphs at the end of Section 2). We generalized this result to graphs in which two points are connected by an edge if and only if their distance is equal to d_k, the k-th largest distance determined by the point set.

Theorem 4 ([9])**.** *For any $k \geq 1$ and $d \geq 2$, there exists a constant $c(d, k)$ satisfying the following condition. Any finite set P of points in $\mathbb{R}^d$ can generate at most $c(d, k)$ regular d-simplices of edge length d_k.*

The bound on $c(d, k)$ given by our proof is an exponential tower in k, which is most likely very far from being best possible. The best known lower bound is only linear in k. For $d = 2$ and 3, we obtained much better bounds, polynomial in k.

Theorem 5.

1. *Any finite set P of points in $\mathbb{R}^2$ can generate at most $O(k^{8/3})$ equilateral triangles of edge length d_k.*
2. *Any finite set P of points in $\mathbb{R}^3$ can generate at most $O(k^{17/3})$ regular tetrahedra of edge length d_k.*

If the points are in convex position in the plane, we can determine the correct order of magnitude of the maximum number of large equilateral triangles.

Theorem 6. *The maximum number of equilateral triangles of side length d_k generated by a finite set of points in convex position in the plane is $\Theta(k)$.*

It would be interesting to determine the exact value of this maximum. In the present note, we establish only Theorem 6.

4 Proof of Theorem 6

The proof uses two lemmas. The first one is due to Erdős, Lovász, and Vesztergombi [5].

Lemma 7. *If a, b, c, d are four distinct vertices of a convex polygon listed in clockwise order such that $|bc| = |ad| = d_k$ for some k, then either between a and b or between c and d there are at most $2k - 3$ other vertices of the polygon.*

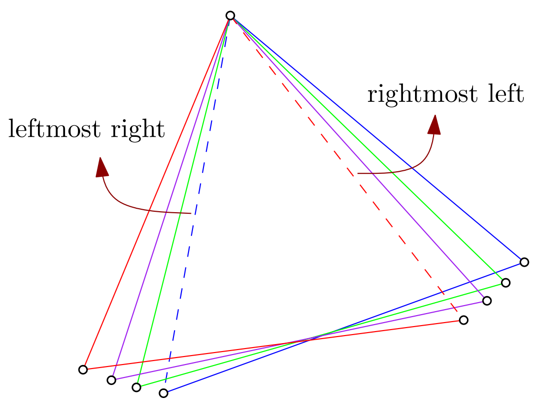

Figure 2: Leftmost right and rightmost left edges with respect to a vertex.

It was proved by Pach and Pinchasi [12] that vertices of a convex n-gon determine at most $\lfloor 2(n-1)/3 \rfloor$ congruent equilateral triangles. Here we give a slightly generalized version of this result, which can be obtained by the same proof technique.

Lemma 8. *Given a convex n-gon with m marked vertices, the number of unit equilateral triangles with at least two marked vertices is at most $2m$.*

Proof. If xyz is a clockwise-oriented unit triangle spanned by the polygon, then xy is said to be a *left edge with respect to x* and a *right edge with respect to y*. It is called the *rightmost left edge with respect to x* if there is no left edge of a unit triangle that can be obtained from xy by a clockwise rotation around x through an angle smaller than π. The *leftmost right edges* are defined analogously (Figure 2).

It is shown in [12] that each edge xy of a unit triangle, which is left with respect to x and right with respect to y in the given triangle, is either the rightmost left edge with respect to x or the leftmost right edge with respect to y.

For each marked vertex, we put a star both on its rightmost left and on its leftmost right edge. Some edges might be eligible for getting stars with respect to their both endpoints: in that case, they will be assigned two stars. In total, we added at most $2m$ stars. On the other hand, at least one edge from every unit triangle that has at least two marked vertices must have a star, namely the edge whose both endpoints are marked. Any edge can be shared by at most two triangles. Note that the edges that are shared by two triangles are both left and right for both of their endpoints. Therefore, they received two stars, and we can conclude that the number of triangles is at most the number of stars, that is, at most $2m$. □

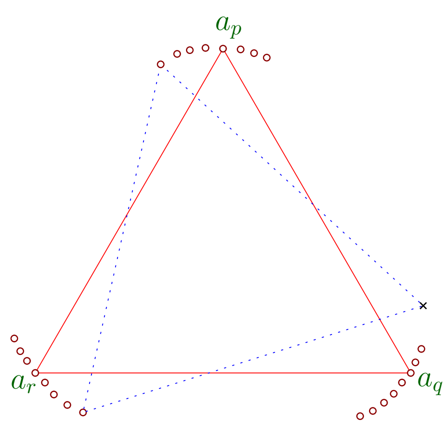

Figure 3: Illustration for the proof of Theorem 6.

Proof of Theorem 6. Let us start with the upper bound. Denote the given points by $a_1, \ldots, a_n$, according to the order in which they appear on the boundary of the convex hull. Let $a_p a_q a_r$ be an equilateral triangle of side length d_k. Let the vertices $a_{p-2k+2}, \ldots, a_p, \ldots, a_{p+2k-2}$, $a_{q-2k+2}, \ldots, a_q, \ldots, a_{q+2k-2}$, $a_{r-2k+2}, \ldots, a_r, \ldots, a_{r+2k-2}$ be marked

(Figure 3). We have at most $12k - 6$ marked vertices and, by Lemma 7, any equilateral triangle of side length d_k has at most one non-marked vertex. Hence, by Lemma 8, we can conclude that the total number of triangles is at most $24k - 12$.

An easy linear lower bound with k triangles is given by the following construction: take a unit equilateral triangle abc and consider $k - 1$ copies of it, obtained by rotations about a through angles $\epsilon, 2\epsilon, \ldots, (k-1)\epsilon$, for a small $\epsilon > 0$. □

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