How Many Potatoes Are in a Mesh?*

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Abstract. We consider the combinatorial question of how many convex polygons can be made at most by using the edges taken from a fixed triangulation of n vertices. For general triangulations, there can be exponentially many: $\Omega(1.5028^n)$ and $O(1.62^n)$ in the worst case. If the triangulation is fat (every triangle has its angles lower-bounded by a constant $\delta > 0$), then there can be only polynomially many: $\Omega(n^{\frac{1}{2}\lfloor \frac{2\pi}{\delta} \rfloor})$ and $O(n^{\lceil \frac{\pi}{\delta} \rceil})$. If we count convex polygons with the additional property that they contain no vertices of the triangulation in their interiors, we get the same exponential bounds in general triangulations, and $\Omega(n^{\lfloor \frac{2\pi}{3\delta} \rfloor})$ and $O(n^{\lfloor \frac{2\pi}{3\delta} \rfloor})$ in fat triangulations.

1 Introduction

It is a common task in combinatorial geometry to give lower and upper bounds for the number of occurrences of a certain subconfiguration in a geometric structure. Well-known examples are the number of vertices in the lower envelope or single face in an arrangement of line segments, the number of triangulations that have a given set of points as their vertices, etc. [10].

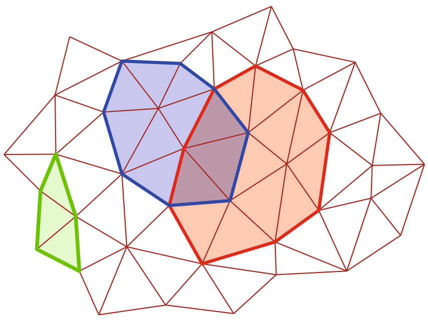

Fig. 1. A mesh M. Three convex polygons that respect M are marked.

In this paper we analyze how many convex polygons (potatoes) can be constructed by taking unions of triangles from a fixed triangulation (mesh) M with n vertices. Equivalently, we analyze how many convex polygon boundaries can be made using the edges of a fixed triangulation, see Figure 1. For general triangulations there can be exponentially many. However, the lower-bound examples use many triangles with very small angles. When $n \to \infty$, the smallest angles tend to zero. To understand if this is necessary, we also study the number of convex polygons in a triangulation, where all angles are bounded from below by a fixed constant. It turns out that the number of convex polygons is polynomial in this case. We also study the same questions when the convex polygons cannot have vertices of M interior to them (carrots). This is the same as requiring that the submesh bounded by the convex polygon is outerplanar.

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Table 1. Results in this paper; open spaces are directly implied by other bounds

input mesh	output vegetable	lower bound	upper bound	source
general	fat carrots	$\Omega(1.5028^n)$		Section 3
general	any potato		$O(1.62^n)$	Section 3
δ-fat	fat potatoes	$\Omega(n^{\frac{1}{2}\lfloor\frac{2\pi}{\delta}\rfloor})$		Section 4
δ-fat	any potato		$O(n^{\lceil\frac{\pi}{\delta}\rceil})$	Section 4
δ-fat	fat carrots	$\Omega(n^{\lfloor\frac{2\pi}{3\delta}\rfloor})$		Section 5
δ-fat	any carrot		$O(n^{\lfloor\frac{2\pi}{3\delta}\rfloor})$	Section 5
compact fat	any carrot	$\Omega(n^2)$	$O(n^2)$	Full version
compact fat	fat carrots	$\Omega(n)$	$O(n)$	Full version

Related Work. This paper is motivated by the *potato peeling problem*: Find a maximum area convex polygon whose vertices and edges are taken from the triangulation of a given point set [2] or a given polygon [4,7].

In computational geometry, *realistic input models* have received considerable attention in the last two decades. By making assumptions on the input, many computational problems can be solved provably faster than what is possible without these assumptions. One of the early examples concerned fat triangles: a triangle is δ-fat if each of its angles is at least δ, for some fixed constant $\delta > 0$. Matousek *et al.* [8] show that the union of n δ-fat triangles has complexity $O(n \log\log n)$ while for n general triangles this is $\Omega(n^2)$. As a consequence, the union of fat triangles can be computed more efficiently as well.

In [1,5,6,9], *fat triangulations* were used as a realistic input model motivated by polyhedral terrains, sometimes with extra assumptions. Fat triangulations are also related to the meshes computed in the area of high-quality mesh generation. The smallest angle of the elements of the mesh is a common quality measure [3]. In graph drawing, an embedded planar straight-line graph is said to have constant *angular resolution* if any two edges meeting at a vertex make at least a constant angle. Hence, fatness and constant angular resolution are the same for triangulations. The original definition of realistic terrains applied to meshes has stronger assumptions than fatness [9]. It also assumes that any two edges in the triangulation differ in length by at most a constant factor, and the outer boundary of the triangulation is a fat convex polygon.

Results. We present lower and upper bounds on the maximum number of convex polygons in a mesh in several settings. The input can be either a *general* mesh, a *fat* mesh (where every angle of each triangle is at least δ), or a *compact fat* mesh (where additionally, the ratio between the shortest and longest edge is at most ρ). The output can be either a *potato* (general convex submesh) or a *carrot* (outerplanar convex submesh, that is, one that contains no vertex of the underlying mesh in its interior), and each can additionally be required to be *fat* (where the ratio between the largest inscribed disk and the smallest containing disk is at most γ). Table 1 summarizes our results. Note that ρ and γ do not show up; the bounds hold for any constant values of ρ and γ.

2 Preliminaries

A *mesh* is a plane straight-line graph with a finite set of vertices, such that all bounded faces are triangles, the interiors of all triangles are disjoint and the intersection of any pair of triangles is either a vertex or a shared edge. We also denote the set of vertices of a graph G by $V(G)$ and the set of edges by $E(G)$, and say the *size* of G is $n = |V(G)|$. We say a mesh M is *maximal* if its triangles completely cover the convex hull of its vertices.[1] A polygon P is said to *respect* a graph G if all of its edges belong to G.

We assume a mesh M is given. We call M δ*-fat*, for some $\delta \in (0, \frac{2}{3}\pi]$, if every angle of every triangle of M is at least δ.

Let $S = [0, 2\pi)$. We define cyclic addition and subtraction $(+, -) : S \times \mathbb{R} \to S$ in the usual way, modulo 2π. We call the elements of S *directions* and implicitly associate an element $s \in S$ with the vector $(\sin s, \cos s)$.

3 Potatoes in General Meshes

Lower Bound. Let Q be a set of m points evenly spaced on the upper half of a circle. Assume $m = 2^k + 1$ for some integer k, and let the points be $v_0, \ldots, v_{m-1}$, clockwise. Let M consist of the convex hull edges, then connect v_0 and v_{m-1} to $v_{(m-1)/2}$, and recursively triangulate the subpolygons by always connecting the furthest pair to the midpoint. Figure 2 illustrates the construction.

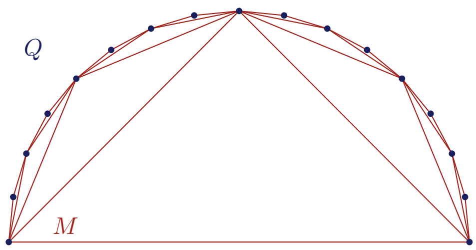

Fig. 2. A set Q of n points on a half-circle, triangulated such that the dual tree is a balanced binary tree

Let $N(k)$ be the number of different convex paths in M from v_0 to v_{m-1}. Then we have $N(k) = 1 + (N(k-1))^2$, $N(0) = 1$, because we can combine every path from v_0 to $v_{(m-1)/2}$ with every path from $v_{(m-1)/2}$ to v_{m-1}, and the extra path is $\overline{v_0, v_{m-1}}$ itself. Using this recurrence, we can relate the number m of vertices used to the number $P(m)$ of convex paths obtained: $P(3) = 2$; $P(5) = 5$; $P(9) = 26$; $P(17) = 677$; etc.

Now we place n points evenly spaced on the upper half of a circle. We triangulate $v_0, \ldots, v_{16}$ as above, and also $v_{16} \ldots, v_{32}$, and so on. We can make $n/16$ groups of 17 points where the first and last point of each group are the same. Each group is triangulated to give 677 convex paths; the rest is triangulated arbitrarily. In total we get $677^{n/16} = \Omega(1.5028^n)$ convex paths from v_0 to v_{n-1}. We omit the one from v_0 directly to v_{n-1}, and use this edge to complete every convex path to a convex polygon. The number of convex polygons is $\Omega(1.5028^n)$.[2]

[1] A maximal mesh is also called a *triangulation*.

[2] We can, of course, make larger groups of vertices to slightly improve the lower bound, but this does not appear to affect the given 4 significant digits.

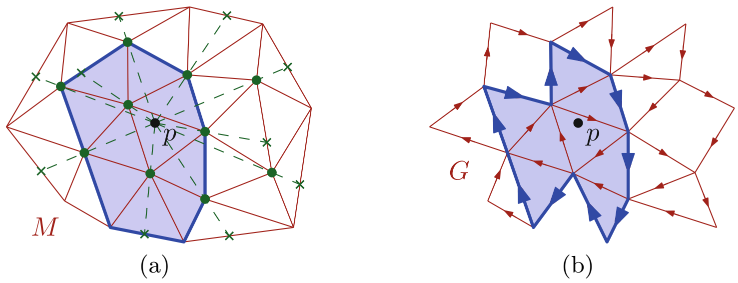

Fig. 3. (a) We project each interior vertex of M from p onto the next edge. An example potato is marked in blue. (b) The graph G obtained by removing the marked edges and orienting the others around p. The potato becomes a cycle.

Theorem 1. *There exists a mesh M with n vertices such that the number of convex polygons that respect M is $\Omega(1.5028^n)$. This is true even if M is the Delaunay triangulation of its vertices.*

Upper Bound. First, fix a point p inside some triangle of M, not collinear with any pair of vertices of M. We count only the convex polygons that contain p for now.

For every vertex v of M, let e_v be the first edge of the mesh beyond v that is hit by a ray from p and through v. Let G be the graph obtained from M by removing all such edges e_v, $v \in V(M)$. Figure 3 shows an example. We turn G into a directed graph by orienting every edge such that p lies to the left of its supporting line. We are interested in the number of simple cycles that respect G. Note that G has exactly $2n-3$ edges, since every vertex not on the convex hull causes one edge to disappear.

Lemma 1. *The number of convex polygons in M that have p in their interior is bounded from above by the number of simple cycles in G.*

Proof. With each convex polygon, we associate a cycle by replacing any edges e_v that were removed by the two edges via v, recursively. This results in a proper cycle because the convex polygon was already a monotone path around p, and this property is maintained. Each convex polygon results in a different cycle because the angle from the vertices of e_v via v is always concave. □

Observation 1. *The complement of the outer face of G is star-shaped with p in its kernel.*

Observation 2. *Let e be an edge on the outer face of G from u to v. Then u has outdegree 1, or v has indegree 1 (or both).*

If $F \subset E(G)$ is a subset of the edges of G, we also consider the subproblem of counting all simple cycles in G that use all edges in F, the *fixed* edges. For a triple (M, G, F), we define the *potential* ρ to be the number of vertices of M (or G) minus the number of edges in F, i.e., $\rho(M, G, F) = |V(G)| - |F|$. Clearly, the potential of a subproblem is an upper bound on the number of edges that can still be used in any simple cycle.

We will now show that the number of cycles in a subproblem can be expressed in terms of subproblems of smaller potential. Let $Q(k)$ be the maximum number of simple cycles in any subproblem with potential k.

Lemma 2. *The function* $Q(\cdot)$ *satisfies* $Q(k) \le Q(k-1) + Q(k-2),\ Q(0) = Q(1) = 1.$

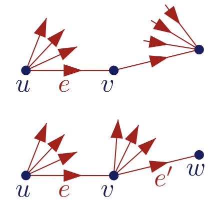

Fig. 4. Two cases for e

Proof. Let (M, G, F) be a subproblem and let $k = \rho(M, G, F)$. If $k = 1$ then $|F| = |V(G)| - 1$, so the number of fixed edges on the cycle is one less than the number of vertices available. Therefore the last edge is also fixed, if any cycle is possible. If $k = 0$, all edges are fixed.

For the general case, suppose all edges on the outer face of G are fixed. Then there is only one possible cycle. If any vertex on the outer face has degree 2 and only one incident edge fixed, we fix the other incident edge too. Suppose there is at least one edge, $e = \overline{uv}$, on the outer face that is not fixed. By Observation 2, one of its neighbors must have degree 1 towards e. Assume without loss of generality that this is v. We distinguish two cases, see Figure 4.

(i) The degree of v is 2. Any cycle in G either uses v or does not use v. If it does not use v we have a subproblem of potential $k - 1$. If it uses v, it must also use its two incident edges, so we can include these edges in F to obtain a subproblem of potential $k-2$. So, the potential $\rho(M, G, F) \le Q(k-1) + Q(k-2)$.

(ii) The degree of v is larger than 2. Any cycle in G either uses e or does not use e. If it uses e, we can add e to F to obtain a subproblem of potential $k - 1$. If it does not use e, then consider v and the edge $e' = \overline{vw}$ that leaves v on the outer face. Since v has indegree 1 but total degree greater than 2, it must have outdegree greater than 1. Therefore, by Observation 2, w must have indegree 1. Therefore, w will not be used by any cycle in G that does not use e, and we can remove v and w to obtain a smaller graph. We also remove all incident edges; if any of them was fixed we have no solutions. We obtain a subproblem of potential $k - 2$ in this case. Again, the potential $\rho(M, G, F) \le Q(k-1) + Q(k-2)$. □

This expression grows at a rate of the root of $x^2 - x - 1 = 0$, which is approximately 1.618034.

Because every convex polygon must contain at least one triangle of M, we just place p in each triangle and multiply the bound by $2n$. Since 1.62 is a slight overestimate (by rounding) of the root, we can ignore the factor $2n$ in the bound.

Theorem 2. *Any mesh* M *with* n *vertices has* $O(1.62^n)$ *convex polygons that respect* M*.*

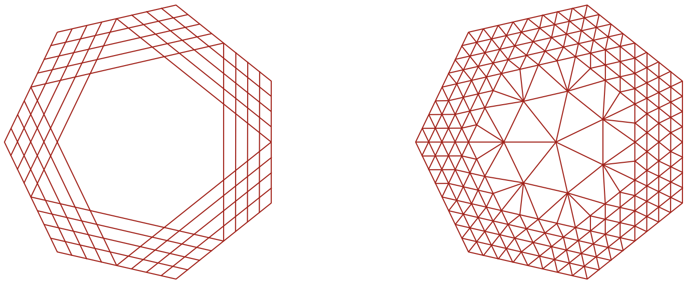

Fig. 5. (a) Essential part of the construction, allowing l^k convex polygons. (b) Final mesh.

4 Potatoes in Fat Meshes

Lower Bound. Let $k = \lfloor \frac{2\pi}{\delta} \rfloor$, and let $l = \sqrt{\frac{n}{2k}}$. Let Q be a regular k-gon, and for each edge e of Q consider the intersection point of the supporting lines of the neighboring edges. Let Q' be a scaled copy of Q that goes through these points. Now, consider a sequence $Q = Q_1, Q_2, \ldots, Q_l = Q'$ of l scaled copies of Q such that the difference in the radii of consecutive copies is equal. We extend the edges of each copy until they touch Q'. Figure 5(a) illustrates the construction.[3]

Observation 3. *The constructed graph has at least l^k different convex polygons.*

We now add vertices and edges to build a δ-fat mesh. We use $\binom{l-1}{2}$ more vertices per sector, placing $l - i$ vertices on each edge of Q_i to ensure that all angles are bounded by δ. We need $O(lk)$ vertices to triangulate the interior using some adaptive mesh generation method. The final mesh can be seen in Figure 5(b). The construction uses $\frac{3}{2}kl^2 + O(kl)$ vertices, and since we have $l = \sqrt{\frac{n}{2k}}$, there are $\frac{3}{2}kl^2 + O(kl) = \frac{3}{2}k\frac{n}{2k} + O(k\sqrt{\frac{n}{2k}}) = \frac{3}{4}n + O(\sqrt{nk}) \leq n$ vertices in total.

Observe that the triangles of the outer ring are Delaunay triangles. The inner part can also be triangulated with Delaunay triangles, since the Delaunay triangulation maximizes the smallest angle of any triangle.

Theorem 3. *There exists a δ-fat mesh M of size n such that the number of convex polygons that respect M is $\Omega(n^{\frac{1}{2}\lfloor \frac{2\pi}{\delta} \rfloor})$. This is true even if M is required to be the Delaunay triangulation of its vertices.*

Upper Bound. We consider paths in M that have roughly consistent directions.

Lemma 3. *Let $u, v \in V(M)$ be two vertices, and let $c, d \in S$ be two directions such that $d - c \leq 2\delta$. Then there is at most one convex path in M from u to v that uses only directions in $[c, d)$.*

[3] Our lower bound constructions use collinear points. We show in the full version that this is not essential, and the same bounds apply to "strictly convex" potatoes.

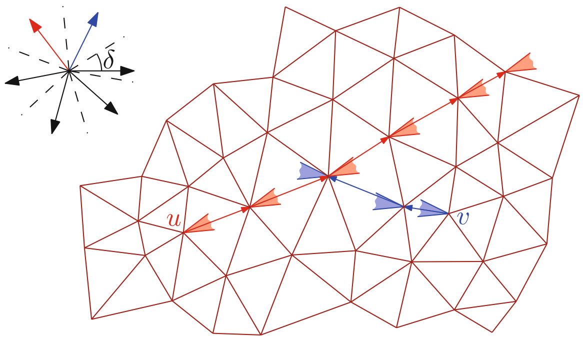

Fig. 6. Two vertices u and v that need to be extreme in two directions that differ by at most 2δ (indicated by red and blue) define a unique potential convex chain since there can be at most one edge in each sector

Proof. Let $m = c + \frac{1}{2}(d - c)$ be the direction bisecting c and d. Because M is δ-fat, for any vertex in $V(M)$ there is at most one incident edge with outgoing direction in $[c, m)$, and also at most one with direction in $[m, d)$. Because the path needs to be convex, it must first use only edges from $[c, m)$ and then switch to only edges from $[m, d)$. We can follow the unique path of edges with direction in $[c, m)$ from u and the unique path of edges with direction in $[m + \pi, d + \pi)$ from v. If these paths intersect, the concatenation may be a unique convex path from u to v as desired (clearly, the path is not guaranteed to be convex, but for an upper bound this does not matter). Figure 6 illustrates this. □

Given a convex polygon P that respects M, a vertex v of P is *extreme* in direction $s \in S$ if there are no other vertices of P further in that direction, that is, if P lies to the left of the line through v with direction $s + \frac{1}{2}\pi$.

Let $\Gamma_\delta = \{0, 2\delta, 4\delta, \ldots, 2\pi\}$ be a set of directions. As an easy corollary of Lemma 3, the vertices of a convex polygon P respecting M that are extreme in the directions of Γ_δ uniquely define P. There are at most n choices for each extreme vertex, so the number of convex polygons is at most $n^{|\Gamma_\delta|}$. Substituting $|\Gamma_\delta| = \lceil \frac{\pi}{\delta} \rceil$ we obtain the following theorem.

Theorem 4. *Any δ-fat mesh M of size n has at most $O(n^{\lceil \frac{\pi}{\delta} \rceil})$ convex polygons that respect M.*

5 Carrots in Fat Meshes

Recall that carrots are potatoes that have no interior vertices from the mesh. So we expect fewer carrots than potatoes. However, our lower bound construction for general meshes only has potatoes that are also carrots. In this section we therefore consider carrots in fat meshes.

Lower Bound. Let $k = \lfloor 2\pi/3\delta \rfloor$, and consider a regular k-gon Q. On each edge of Q, we place a triangle with angles δ, 2δ, and $\pi - 3\delta$. Then, we subdivide

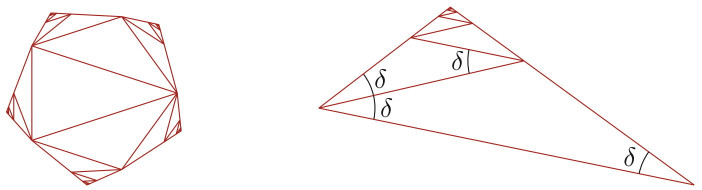

Fig. 7. (a) An example of a δ-fat mesh obtained from a k-gon ($k = 5$), which has $\Omega(n^k)$ carrots. (b) A tower of δ-δ-$(\pi - 2\delta)$ triangles.

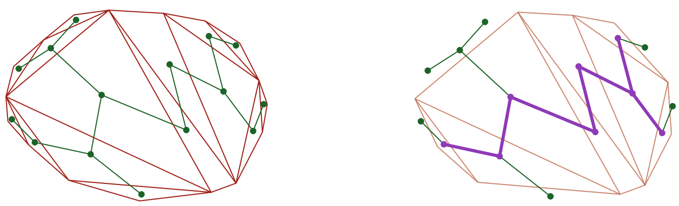

Fig. 8. (a) A carrot and its dual tree. (b) The skeleton (shown bold) of the dual tree is the spanning tree of all vertices of degree 2.

each such triangle into $\frac{n-k}{k}$ smaller triangles with angles δ, δ, and $\pi - 2\delta$, as illustrated in Figure 7(b). Finally, we triangulate the internal region of Q in any way we want, giving a mesh M.

Lemma 4. *M is convex, δ-fat, and contains $\Omega(n^{\lfloor \frac{2\pi}{3\delta} \rfloor})$ carrots.*

Proof. M is convex because $\delta + 2\delta \leq \frac{2\pi}{k}$. Every angle in the triangles outside Q is at least δ, and the angles in the interior of Q are multiples of $\frac{\pi}{k} > \delta$. Therefore, every connected subset of M is a carrot. The dual tree T of M has a central component consisting of k vertices, and then k paths of length $\frac{n}{k} - 1$. Hence, the number of subtrees of T is at least $(\frac{n}{k} - 1)^k$, which is $\Omega(n^{\lfloor \frac{2\pi}{3\delta} \rfloor})$. □

Theorem 5. *There exists a δ-fat mesh M of size n such that the number of convex outerplanar polygons that respect M is $\Omega(n^{\lfloor \frac{2\pi}{3\delta} \rfloor})$.*

Upper Bound. We will next show that given any δ-fat mesh M, the number of carrots that respect M can be at most $O(n^{\lfloor \frac{2\pi}{3\delta} \rfloor})$.

Consider any carrot. We inspect the dual tree T of the carrot and make some observations. Each node of T is either a *branch node* (if it has degree 3), a *path node* (if it has degree 2), or a *leaf* (if it has degree 1). Path nodes have one edge on the boundary of the carrot, and leaves have two edges on the boundary of the carrot. Figure 8(a) shows an example.

Fig. 9. (a) Every leaf gives rise to a turning angle of 2δ. (b) Every leaf that is an only child gives rise to a turning angle of 3δ.

Observation 4. *Let v be a leaf node of T. The turning angle between the two external edges of v is at least 2δ.*

Proof. The triangle for node v is δ-fat, so all three angles are $\geq \delta$. Therefore, the angles are $\leq \pi - 2\delta$, and the turning angles are $\geq 2\delta$ (Figure 9(a)). □

Observation 5. *Let v be a leaf node of T and u a path node adjacent to v. The turning angle between the external edge of the triangle for u and the furthest external edge of the triangle for v is at least 3δ.*

Proof. Consider the quadrilateral formed by the two triangles of u and v. The edge in M separating u from the rest of T has two δ-fat triangles incident to one of its endpoints, and one to its other endpoint. This means that the turning angle between the edges in the observation is $\geq 3\delta$ (Figure 9(b)). □

By Observation 4, the number of leaves in a carrot is bounded by $\lfloor \frac{\pi}{\delta} \rfloor$, and therefore, also the number of branch nodes is bounded by $\lfloor \frac{\pi}{\delta} \rfloor - 2$. However, the number of path nodes can be unbounded. Consider subtree S of T that is the spanning tree of all the path nodes. We call S the *skeleton* of the carrot. Figure 8(b) shows an example. By Observation 5, the number of leaves of S is bounded by $\lfloor \frac{2\pi}{3\delta} \rfloor$.

We will charge the carrot to the set of leaves of S, and we will argue that every set of $\lfloor \frac{2\pi}{3\delta} \rfloor$ triangles in M is charged only constantly often (for constant δ).

Observation 6. *Let Δ be any set of triangles of M. If there exists a carrot that contains all triangles in Δ, then there is a unique smallest such carrot.*

Lemma 5. *Let Δ be any set of triangles in M. The number of carrots that charge Δ is at most $2^{\lfloor \frac{2\pi}{\delta} \rfloor}$.*

Proof. Consider the tree S that is the dual of the unique smallest carrot that contains Δ, as per Observation 6. Any carrot that charges Δ has S as its skeleton. First, we argue that the set of path nodes in any carrot that charges Δ is a subset of S. Indeed, if there was any path node in T outside S, then there would be at least one leaf component of T that is disconnected from S, and there would be an edge outside Δ that gets charged by the carrot of T. Therefore, only branch nodes and leaves can still be added to S to obtain a carrot that charges Δ.

Then, we argue that there are at most $2^{\lfloor \frac{2\pi}{\delta} \rfloor}$ other nodes that can be part of a carrot that charges Δ. We can augment S by adding on components consisting of only k leaves and $k-1$ branch nodes. By Observation 4, each such component consumes a turning angle of $2k\delta$. Therefore, they can only be added on edges of S which have a cap angle of at least $2k\delta$. Therefore, there can be at most $2\pi/\delta$ potential leaves, leading to $2^{\lfloor \frac{2\pi}{\delta} \rfloor}$ choices.[4] □

Theorem 6. *Any δ-fat mesh M of size n has at most $O(n^{\lfloor \frac{2\pi}{3\delta} \rfloor})$ convex outerplanar polygons that respect M.*

When the mesh is not only fat, but the edge length ratio is also bounded by a constant, we can prove better bounds. We call such meshes *compact fat.* We state the results here but defer the proofs to the full version.

Theorem 7. *Any compact fat mesh M of size n has at most $O(n)$ convex fat outerplanar polygons that respect M.*

Theorem 8. *Any compact fat mesh M of size n has at most $O(n^2)$ convex outerplanar polygons that respect M.*

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[4] Not all potential leaves can be chosen independently, but we ignore this issue since the factor is dominated by the dependency on n anyway.

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