Note

Convexly independent subsets of the Minkowski sum of planar point sets

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Abstract

Let P and Q be finite sets of points in the plane. In this note we consider the largest cardinality of a subset of the Minkowski sum $S \subseteq P \oplus Q$ which consist of convexly independent points. We show that, if $|P| = m$ and $|Q| = n$ then $|S| = O(m^{2/3}n^{2/3} + m + n)$.

1 Introduction

In connection with a class of convex combinatorial optimization problems (Onn and Rothblum, 2004), Halman et al. (2007) raised the following question. Given a set X of n points in the plane, what is the maximum number of pairs that can be selected from X so that the midpoints of their connecting segments are *convexly independent*, that is, they form the vertex set of a convex polygon? In the special case when the elements of X themselves are convexly independent, they found a linear upper bound, $5n-6$, on this quantity. They asked whether there exists a subquadratic upper bound in the general case. In this note, we answer this question in the affirmative by establishing an upper bound of $O(n^{4/3})$.

We first reformulate the question in a slightly more general form. Let P and Q be sets of size m and n in the plane. The *Minkowski sum* of P and Q is $P \oplus Q = \{p + q \mid p \in P,\, q \in Q\}$.

What is the maximum size of a convexly independent subset of $P \oplus Q$?

More precisely, we would like to estimate the function $M(m, n)$, which is the largest cardinality of a convexly independent set S, which is a subset of the Minkowski sum of some planar point sets P and Q with $|P| = m$ and $|Q| = n$.

Notice that the set of all midpoints of the connecting segments of an n-element set P can be expressed as $\frac{1}{2}(P \oplus P)$, so that $M(n, n)$ is an upper bound on the quantity studied by Halman et al.

Let S be a convexly independent subset of $P \oplus Q$. Consider the bipartite graph G on the vertex set $P \cup Q$, in which $p \in P$ and $q \in Q$ are connected by an edge if and only if $p + q \in S$. It is easy to check that G cannot contain $K_{2,3}$ as a subgraph. Applying the *forbidden subgraph theorem* (Kővári et al., 1954), see also (Pach and Agarwal, 1995), it follows that $|S| = O(\sqrt{m} \cdot n + m)$.

Our next result provides a better bound.

Theorem 1. *Let P and Q be two planar point sets with $|P| = m$ and $|Q| = n$. For any convexly independent subset $S \subseteq P \oplus Q$, we have $|S| = O(m^{2/3}n^{2/3} + m + n)$.*

2 Proof of Theorem 1

We reduce the problem to a *point-curve incidence problem* in the plane. A closed set $K \subseteq \mathbb{R}^2$ is *strictly convex*, if for each $a, b \in K$ the interior of the line-segment $\mathrm{conv}(\{a, b\})$ is contained in the interior of K. A closed curve C is *strictly convex* if it is the boundary of a strictly convex set. Consider now n translated copies $C + t_1, \ldots, C + t_n$ of C, and m points $p_1, \ldots, p_m$. Let $I(m, n)$ denote the maximum number of point-curve incidences which occur in such a configuration. Notice that $C + t_i$ and $C + t_j$ intersect in at most two points for $i \neq j$. Furthermore, for any two distinct points p_μ and p_ν, there exist at most two curves $C + t_i$ incident to both p_μ and p_ν. We can apply the following well known upper bound on the number $I(m, n)$ of incidences between m points and n "well-behaved" curves with the above properties, see (Pach and Sharir, 1998).

$$I(m, n) = O(m^{2/3}n^{2/3} + m + n). \tag{1}$$

Thus, to establish Theorem 1, it remains to prove

Theorem 2. *For any positive integers m and n, we have $M(m, n) \leqslant I(m, n)$.*

Proof. Let $P = \{p_1, \ldots, p_m\}$, $Q = \{q_1, \ldots, q_n\}$, and assume that S is a convexly independent subset of $P \oplus Q$. Clearly, there is a strictly convex closed curve C passing through all points in S. Consider the n translates $C - q_1, \ldots, C - q_n$ of C. Count the number of incidences between these curves and the elements of P. Notice that if the point $p + q$ belongs to S, then p is incident to $C - q$. Since no two distinct points $p_1 + q_1 \neq p_2 + q_2 \in S$ are associated with the same incidence, the result follows. ☐

Unit distances

Theorem 1 can also be deduced from the known upper bounds on the number of unit-distance pairs induced by n points in a normed (Minkowski) plane. For this, notice that one can replace C by a centrally symmetric strictly convex curve C' such that the number I' of incidences between the curves $C' - q_1, \dots, C' - q_n$ and the points in P is at least half of the number I of incidences between the curves $C - q_1, \dots, C - q_n$ and the points in P. The curve C' defines a *norm*, and thus a *metric*, in the plane, with respect to which the unit circle is a translate of C'. Therefore, I' can be bounded from above by the number of unit-distance pairs between the set of centers of the curves $C' - q_1, \dots, C' - q_n$ and the elements of P, which is known to be $O(m^{2/3}n^{2/3} + m + n)$.

In particular, for $m = n$, this number cannot exceed the maximum number $u(2n)$ of unit-distance pairs in a set of $2n$ points in a normed plane with a strictly convex unit circle. It is known that $u(2n) = O(n^{4/3})$ (see e.g. (Brass, 1996)), and a gridlike construction shows that this bound can be attained for certain norms (Brass, 1998; Valtr, 2005). Note that in the Euclidean norm, the number of unit-distance pairs induced by n points is $ne^{\Omega(\log n/ \log\log n)}$, and this estimate is conjectured to be not far from best possible (Erdős, 1946).

The question arises whether any of the examples establishing the tightness of the upper bounds on $I(m, n)$ and $u(n)$ can be used to show that Theorem 1 is also optimal. Unfortunately, in all known constructions, most elements of $P \oplus Q$ can be written in the form $p + q$ ($p \in P, q \in Q$) in many different ways. Therefore, any element of a convexly independent subset of $P \oplus Q$ may be associated with several incidences between a curve $C - q$ and a point of P. This suggests that the maximum size of a convexly independent subset of $P \oplus Q$ can be much smaller than $I(m, n)$. For $m = n$, we do not know any example for which $P \oplus Q$ has a convexly independent subset with a superlinear number of elements.

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