

On the number of edges of separated multigraphs

Jacob Fox^{1*}, János Pach^{2**}, and Andrew Suk^{3***}

¹ Stanford University, Stanford, CA, USA

jacobfox@stanford.edu

² Rényi Institute, Budapest and MIPT, Moscow

pach@cims.nyu.edu

³ University of California at San Diego, La Jolla, CA, USA

asuk@ucsd.edu

Abstract. We prove that the number of edges of a multigraph G with n vertices is at most $O(n^2 \log n)$, provided that any two edges cross at most once, parallel edges are noncrossing, and the lens enclosed by every pair of parallel edges in G contains at least one vertex. As a consequence, we prove the following extension of the Crossing Lemma of Ajtai, Chvátal, Newborn, Szemerédi and Leighton, if G has $e \geq 4n$ edges, in any drawing of G with the above property, the number of crossings is $\Omega\left(\frac{e^3}{n^2 \log(e/n)}\right)$. This answers a question of Kaufmann *et al.* and is tight up to the logarithmic factor.

Keywords: Multigraphs · Lenses · Crossing Lemma.

1 Introduction

A *topological graph* is a graph drawn in the plane such that its vertices are represented by points, and the edges are represented by simple continuous arcs connecting the corresponding pairs of points. In notation and terminology, we do not distinguish between the vertices and the points representing them and the edges and the arcs representing them. The edges are allowed to intersect, but they cannot pass through any vertex other than their endpoints. If two edges share an interior point, then they must properly *cross* at that point, *i.e.*, one edge passes from one side of the other edge to its other side.

A *multigraph* is a graph in which two vertices can be joined by several edges. Two edges that join the same pair of vertices are called *parallel*.

According to the *crossing lemma* of Ajtai, Chvátal, Newborn, Szemerédi [1] and Leighton [4], every topological graph G with n vertices and $e > 4n$ edges

* Supported by a Packard Fellowship and by NSF award DMS-1855635.

** Supported by NKFIH grants K-176529, KKP-133864, Austrian Science Fund Z 342-N31, Ministry of Education and Science of the Russian Federation MegaGrant No. 075-15-2019-1926, ERC Advanced Grant “GeoScape.”

*** Supported by an NSF CAREER award, NSF award DMS-1952786, and an Alfred Sloan Fellowship.

has at least $c \frac{e^3}{n^2}$ edge crossings, where $c > 0$ is an absolute constant. In notation, we have

$$\text{cr}(G) \geq c \cdot \frac{e^3}{n^2}. \quad (1)$$

In a seminal paper which was an important step towards the solution of Erdős's famous problem on distinct distances [2], Székely [10] generalized the crossing lemma to multigraphs: for every topological multigraph G with n vertices and $e > 4n$ edges, in which the *multiplicity* of every edge is at most m , we have

$$\text{cr}(G) \geq c \frac{e^3}{mn^2}. \quad (2)$$

As the maximum multiplicity m increases, (2) gets weaker. However, as was shown in [7] and [3], under certain special conditions on the multigraphs, the inequality (1) remains true, independently of m . Some related results were established in [8]. In all of these papers, one of the key elements of the argument was to find an analogue of Euler's theorem for the corresponding classes of "nearly planar" multigraphs.

Throughout this paper, we consider only *single-crossing* topological multigraphs, *i.e.*, we assume that any two edges cross *at most once*. Hence, two edges that share an endpoint may also have a common interior point. Two edges are said to be *independent* if they do not share an endpoint, and they are called *disjoint* if they are independent and do not cross.

Definition 1. *A multigraph G is called separated if no two parallel edges of G cross, and the "lens" enclosed by them has at least one vertex in its interior.*

It was conjectured in [3] that any separated single-crossing topological multigraph with n vertices has at most $O(n^2)$ edges. The aim of this note is to verify this conjecture apart from a logarithmic factor.

Theorem 1. *The number of edges of a separated single-crossing topological multigraph G on n vertices satisfies $|E(G)| \leq O(n^2 \log n)$.*

Note that in a separated multigraph, any pair of vertices can be connected by at most $n - 1$ edges. This immediately implies the bound $|E(G)| \leq \binom{n}{2}(n - 1) = O(n^3)$.

If we plug in Theorem 1 into the machinery of [3] and [7], a routine calculation gives the following.

Corollary 1. *Every separated single-crossing topological multigraph on n vertices and $e \geq 4n$ edges has at least $c \frac{e^3}{n^2 \log(e/n)}$ crossings, where $c > 0$ is a suitable constant.*

For simplicity, we will assume that a multigraph does not have *loops*. It is easy to see that Theorem 1 also holds for topological multigraphs with loops, assuming that each loop contains a vertex.

Theorem 1 does not remain true if we replace the assumption that G is *single-crossing* by the weaker one that any two edges cross at most *twice*. To see this, let the vertices of G lie on the x -axis: set $V(G) = \{1, 2, \dots, n\}$. Let each edge consist of a semicircle in the upper half-plane and a semicircle below it that meet at a point of the x -axis. More precisely, for any pair of integers $i, j \in V(G)$ with $i < j$, and for any k with $i \leq k < j$, pick a distinct point p_{ikj} in the open interval $(k, k+1)$. Let γ_{ikj} be the union of two semicircles centered at the x -axis: an upper semicircle connecting i to p_{ikj} and a lower one connecting p_{ikj} to j . Let $E(G)$ consist of all arcs γ_{ikj} over all triples $i \leq k < j$. Observe that any two edges of G cross at most twice: once above the x -axis and once below it. No two parallel edges, γ_{ihj} and γ_{ikj} with $h < k$, cross each other, and the region enclosed by them contains the vertex $k \in V(G)$. Therefore, G is a separated topological multigraph with $\sum_{i,j(i < j)} (j - i) = \Omega(n^3)$.

The proof of Theorem 1 is presented in the next section.

All logarithms used in the sequel are of base 2. We omit all floor and ceiling signs wherever they are not crucially important.

2 Proof of Theorem 1

We will need the following simple lemma.

Lemma 1. *Let G be a single-crossing topological graph on n vertices with no parallel edges, in which every pair of independent edges cross. Then we have $|E(G)| \leq 4n$.*

Proof. Let $V(G) = A \cup B$ be a bipartition of the vertex set such that at least half of the edges of G run between A and B . Denote the corresponding bipartite graph by $G(A, B)$. Any pair of independent edges of $G(A, B)$ cross once, that is, an *odd* number of times. Assume without loss of generality that A and B are separated by a horizontal line. By “flipping” one of the half-planes bounded by this line from left to right, we obtain a drawing of $G(A, B)$, in which any pair of independent edges cross an *even* number of times. According to the Hanani-Tutte theorem [11, 9], this implies that $G(A, B)$ is a planar graph. Any bipartite planar graph on $n \geq 3$ vertices has at most $2n - 4$ edges. Therefore, $|E(G)| \leq 2|E(G(A, B))| \leq 4n - 8$.

Proof (Proof of Theorem 1). Let $G = (V, E)$ be a separated single-crossing topological multigraph on n vertices. If two vertices, u and v , are joined by $j > 1$ parallel edges, then they cut the plane into j pieces, one of which is unbounded. The bounded pieces are called *lenses*. Each lens is bounded by two adjacent edges joining a pair of vertices. Let L denote the set of lenses determined by G .

If $|L| \leq \frac{|E(G)|}{2}$, then keeping only one edge between every pair of adjacent vertices, we obtain a simple graph G' whose number of edges satisfies

$$\frac{|E(G)|}{2} \leq |E(G')| \leq \binom{n}{2}.$$

This implies that $|E(G)| < n^2$, and we are done.

From now on, we can and will assume that

$$|L| \geq \frac{|E(G)|}{2}. \quad (3)$$

For any lens $\ell \in L$, let $|\ell|$ denote the number of vertices in the interior of ℓ . For $t = \log n$, we partition L into t parts, $L_1 \cup L_2 \cup \dots \cup L_t$, where $\ell \in L_i$ if and only if $2^{i-1} \leq |\ell| < 2^i$. By the pigeonhole principle, there is an integer $k, 1 \leq k \leq t$, such that

$$|L_k| \geq \frac{|L|}{\log n}. \quad (4)$$

Fix an integer k with the above property, and let $d_k(v)$ denote the number of lenses in L_k that contain vertex v in its interior. Then we have

$$\sum_{v \in V} d_k(v) = \sum_{\ell \in L_k} |\ell| \geq |L_k| 2^{k-1}.$$

Hence, there is a vertex $v \in V$ that lies in the interior of at least $|L_k| \frac{2^{k-1}}{n}$ lenses from L_k . Assume without loss of generality that v is located at the origin o , and let L^o denote the set of lenses in L_k which contain the origin. Hence, we have

$$|L^o| \geq |L_k| \cdot \frac{2^{k-1}}{n}.$$

Combining this with (3) and (4), we obtain

$$|L^o| \geq \frac{|E(G)|}{n \log n} \cdot 2^{k-2}. \quad (5)$$

Let G^o denote the subgraph of G consisting of all vertices and the edges that bound a lens in L^o . Any two vertices of G^o are connected by 0 or 2 edges of G^o .

Now we use the idea of the probabilistic proof of the crossing lemma; see [5]. Let W be a random subset of V in which each vertex is picked independently with probability $p = 2^{-k}$. Let $G^o[W]$ be the subgraph of G^o induced by W . Let $L^o(W)$ denote the set of *empty* lenses in $G^o[W]$ (that is, the set of lenses with empty interiors). For the expected values of $|W|$ and $|L^o(W)|$, we have

$$\mathbb{E}[|W|] = pn$$

and

$$\mathbb{E}[|L^o(W)|] \geq p^2(1-p)^{2k} |L^o|.$$

By linearity of expectation, there is a subset W' of V such that

$$|L^o(W')| - 4|W'| \geq \mathbb{E}[|L^o(W)|] - 4\mathbb{E}[|W|] \geq p^2(1-p)^{2k} |L^o| - 4pn. \quad (6)$$

For each lens in $\ell \in L^o(W')$, we arbitrarily pick one of the two edges enclosing ℓ , and denote the resulting simple topological graph by G' . We now make the following observation.

Lemma 2. *Any two independent edges of G' cross each other.*

Proof. Suppose, for contradiction, that G' has two independent edges, e and e' , which do not cross. Let ℓ and ℓ' be the corresponding empty lenses in $G^o[W']$. Since the interiors of ℓ and ℓ' are empty, neither of them can contain an endpoint of the other. Both of these lenses contain the origin o , which implies that they cannot be disjoint. Therefore, both sides of ℓ must cross both sides of ℓ' , contradicting the choice of e and e' . Here, we used the assumption that G and, hence, G' are single-crossing.

In view of Lemma 2, we can apply Lemma 1 to G' . We obtain $|E(G')| = |L^o(W')| \leq 4|W'|$ and hence by (6) we have $p^2(1-p)^{2k}|L^o| \leq 4pn$. It follows that

$$|L^o| \leq 4p^{-1}(1-p)^{-2k}n.$$

Substituting $p = 2^{-k}$, we get

$$|L^o| \leq 16 \cdot 2^k n.$$

Comparing this with (5), we conclude that

$$|E(G)| \leq O(n^2 \log n).$$

This completes the proof of Theorem 1.

References

1. M. Ajtai, V. Chvátal, M. N. Newborn, and E. Szemerédi: Crossing-free subgraphs, in: *Theory and Practice of Combinatorics*, North-Holland Mathematics Studies **60**, North-Holland, Amsterdam, 9–12, 1982.
2. L. Guth and N. H. Katz: On the Erdős distinct distances problem in the plane, *Annals of Mathematics* **181** (2015), no. 1, 155–190.
3. M. Kaufmann, J. Pach, G. Tóth, and T. Ueckerdt: The number of crossings in multigraphs with no empty lens, in: *Graph Drawing and Network Visualization, Lecture Notes in Comput. Sci.* **11282**, Springer, Cham, 2018, 242–254.
4. T. Leighton: *Complexity Issues in VLSI, Foundations of Computing Series*, MIT Press, Cambridge, 1983.
5. J. Matoušek: *Lectures on Discrete Geometry. Graduate Texts in Mathematics* **212**, Springer-Verlag, New York, 2002.
6. J. Matoušek, Bounded VC-Dimension implies a fractional Helly theorem, *Discrete Comput. Geom.* **31** (2004), 251–255.
7. J. Pach and G. Tóth: A crossing lemma for multigraphs, *Discrete Comput. Geom.* **63** (2020), no. 4, 918–933.
8. J. Pach, G. Tardos, and G. Tóth: Crossings between non-homotopic edges, in: *Graph Drawing and Network Visualization, Lecture Notes in Comput. Sci.* **12590**, Springer-Verlag, Cham, 2020, 359–371.
9. M. Schaefer: Toward a theory of planarity: Hanani-Tutte and planarity variants, *J. Graph Algorithms Appl.* **17** (2013), no. 4, 367–440.

10. L. A. Székely: Crossing numbers and hard Erdős problems in discrete geometry, *Combin. Probab. Comput.* **6** (1997), no. 3, 353–358.
11. W. T. Tutte: Toward a theory of crossing numbers, *J. Combinatorial Theory* **8** (1970), 45–53.