SPHERE-OF-INFLUENCE GRAPHS IN NORMED SPACES

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Dedicated to Károly Bezdek and Egon Schulte on the occasion of their 60th birthdays

Abstract. We show that any k-th closed sphere-of-influence graph in a d-dimensional normed space has a vertex of degree less than $5^d k$, thus obtaining a common generalization of results of Füredi and Loeb (1994) and Guibas, Pach and Sharir (1994).

Toussaint [Tou88] introduced the sphere-of-influence graph of a finite set of points in Euclidean space for applications in pattern analysis and image processing (see [Tou14] for a recent survey). This notion was later generalized to so-called closed sphere-of-influence graphs [HJLM93] and to k-th closed sphere-of-influence graphs [KZ04]. Our setting will be a d-dimensional normed space $\mathcal{N}$ with norm $\|\cdot\|$. We denote the ball with center $c \in \mathcal{N}$ and radius r by $B(c, r)$.

Definition 1. Let $k \in \mathbb{N}$ and let $V = \{c_i : i = 1, \dots, m\}$ be a set of points in the d-dimensional normed space $\mathcal{N}$. For each $i \in \{1, \dots, m\}$, let $r_i^{(k)}$ be the smallest r such that

$$\{j \in \mathbb{N} : j \neq i, \|c_i - c_j\| \leq r\}$$

has at least k elements. Define the *k-th closed sphere-of-influence graph on V* by setting $\{c_i, c_j\}$ an edge whenever $B(c_i, r_i^{(k)}) \cap B(c_j, r_j^{(k)}) \neq \emptyset$.

Füredi and Loeb [FL94] gave an upper bound for the minimum degree of any closed sphere-of-influence graph in $\mathcal{N}$ in terms of a certain packing quantity of the space (see also [MQ94, Sul94].)

Definition 2. Let $\vartheta(\mathcal{N})$ denote the largest cardinality of a subset A of the ball $B(o, 2)$ of the normed space $\mathcal{N}$ such that any two points of A are at distance at least 1, and the origin o is in A.

Füredi and Loeb [FL94] showed that any closed sphere-of-influence graph (that is, in our terminology, a first closed sphere-of-influence graph) in $\mathcal{N}$ has a vertex of degree smaller than $\vartheta(\mathcal{N}) \leq 5^d$. (It is clear that $\vartheta(\mathcal{N})$ is bounded above by the number of balls of radius $1/2$ that can be packed into a ball of radius $5/2$, which is at most 5^d by volume considerations.)

Guibas, Pach and Sharir [GPS94] showed that any k-th closed sphere-of-influence graph in d-dimensional Euclidean space has a vertex of degree at most $c^d k$, for some universal constant $c > 1$. In this note we show the following more precise result, valid for all norms, and generalizing the result of Füredi and Loeb [FL94] mentioned above.

Theorem 3. *Every k-th sphere-of-influence graph on at least two points in a normed space $\mathcal{N}$ has at least two vertices of degree smaller than $\vartheta(\mathcal{N})k \leq 5^d k$.*

We note that the theorem still holds when there are repeated elements.

Corollary 4. *A k-th sphere-of-influence graph on n points in $\mathcal{N}$ has at most $(\vartheta(\mathcal{N})k - 1)n \leq (5^d k - 1)n$ edges.*

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Proof of Theorem 3. Let $V = \{c_1, c_2, \dots, c_m\}$. Relabel the vertices $c_1, c_2, \dots, c_m$ such that $r_1^{(k)} \le r_2^{(k)} \le \dots \le r_m^{(k)}$. We define an auxiliary graph H on V by joining c_i and c_j whenever $\|c_i - c_j\| < \max\{r_i^{(k)}, r_j^{(k)}\}$. Thus, if $\{c_i \colon i \in I\}$ is an independent set in H, then no ball in $\{B(c_i, r_i^{(k)}) \colon i \in I\}$ contains the center of another in its interior. We next bound the chromatic number of H.

Lemma 5. *The chromatic number of H does not exceed k.*

Proof. Note that for each $i \in \{1, \dots, m\}$, the set

$$\{j < i \colon c_i c_j \in E(H)\} = \{j < i \colon \|c_i - c_j\| < r_i^{(k)}\}$$

has less than k elements. Therefore, we can greedily color H in the order $c_1, c_2, \dots, c_m$ by k colors. □

We next show that the degrees of c_1 and c_2 (corresponding to the two smallest values of $r_i^{(k)}$) are both at most $\vartheta(\mathcal{N})k$, which will complete the proof of Theorem 3. We first need the so-called "bow-and-arrow" inequality of [FL94]. For completeness, we include the proof from [FL94].

Lemma 6 (Füredi–Loeb [FL94])**.** *For any two non-zero elements a and b of a normed space,*

$$\left\| \frac{1}{\|a\|} a - \frac{1}{\|b\|} b \right\| \ge \frac{\|a - b\| - |\|a\| - \|b\||}{\|b\|}.$$

Proof. Without loss of generality, we may assume that $\|a\| \ge \|b\| > 0$. Then

$$\begin{aligned} \|a - b\| &= \left\| \|a\| \frac{1}{\|a\|} a - \|b\| \frac{1}{\|b\|} b \right\| \\ &= \left\| \|b\| \left(\frac{1}{\|a\|} a - \frac{1}{\|b\|} b\right) + (\|a\| - \|b\|) \frac{1}{\|a\|} a \right\| \\ &\le \|b\| \left\| \frac{1}{\|a\|} a - \frac{1}{\|b\|} b \right\| + \|a\| - \|b\| . \end{aligned}$$ □

The next lemma is abstracted with minimal hypotheses from [MQ94, Proof of Theorem 6] (see also [FL94, Proof of Theorem 2.1]).

Lemma 7. *Consider the balls $B(v_1, \lambda_1)$ and $B(v_2, \lambda_2)$ in the normed space $\mathcal{N}$, such that $\max\{\lambda_1, \lambda_2\} \ge 1$, $v_1 \notin \operatorname{int}(B(v_2, \lambda_2))$, $v_2 \notin \operatorname{int}(B(v_1, \lambda_1))$ and $B(v_i, \lambda_i) \cap B(o, 1) \ne \emptyset$ $(i = 1, 2)$. Define $\pi \colon \mathcal{N} \to B(o, 2)$ by*

$$\pi(x) = \begin{cases} x & \text{if } \|x\| \le 2, \\ \frac{2}{\|x\|} x & \text{if } \|x\| \ge 2. \end{cases}$$

Then $\|\pi(v_1) - \pi(v_2)\| \ge 1$.

Proof. In terms of the norm, we are given that $\|v_1 - v_2\| \ge \max\{\lambda_1, \lambda_2\} \ge 1$, $\|v_1\| \le \lambda_1 + 1$, and $\|v_2\| \le \lambda_2 + 1$. Without loss of generality, we may assume that $\|v_2\| \le \|v_1\|$.

If $v_1, v_2 \in B(o, 2)$ then $\|\pi(v_1) - \pi(v_2)\| = \|v_1 - v_2\| \ge 1$.

If $v_1 \notin B(o, 2)$ and $v_2 \in B(o, 2)$, then

$$\begin{aligned} \|\pi(v_1) - \pi(v_2)\| &= \left\| 2 \frac{1}{\|v_1\|} v_1 - v_2 \right\| \ge \|v_1 - v_2\| - \left\| v_1 - 2 \frac{1}{\|v_1\|} v_1 \right\| \\ &= \|v_1 - v_2\| - (\|v_1\| - 2) \ge \lambda_1 - (\lambda_1 + 1) + 2 = 1. \end{aligned}$$

If $v_1, v_2 \notin B(o,2)$, then

$$\|\pi(v_1)-\pi(v_2)\| = \left\|2\frac{1}{\|v_1\|}v_1 - 2\frac{1}{\|v_2\|}v_2\right\| \geq 2\frac{\|v_1-v_2\|-\|v_1\|+\|v_2\|}{\|v_2\|} \quad \text{by Lemma 6}$$

$$\geq 2\left(\frac{\lambda_1-(\lambda_1+1)}{\|v_2\|}+1\right) = \frac{-2}{\|v_2\|}+2 \geq -1+2 = 1. \qquad \square$$

We can now finish the proof of Theorem 3. Let $i \in \{1,2\}$, and let $c := c_i$, that is, the radius corresponding to c is the smallest, or second smallest. By Lemma 5 we can partition the set of neighbors of c in the k-th closed sphere-of-influence graph on V into k classes $N_1,\dots,N_k$ so that each N_t is an independent set in H. We may assume that the radius $r_i^{(k)}$ corresponding to c is 1. Then for any $t \in \{1,\dots,k\}$, each ball in $\{B(c_j, r_j^{(k)})\colon c_j \in N_t\}$ intersects $B(c,1)$, and the center of no ball is in the interior of another ball. By Lemma 7, $\{\pi(p-c)\colon p \in N_t\}$ is a set of points contained in $B(o,2)$ with a distance of at least 1 between any two. That is, $|N_t \setminus \operatorname{int}(B(c,1))| \leq \vartheta(\mathcal{N})-1$ for each $t=1,\dots,k$. Since there are at most $k-1$ points in $V \cap \operatorname{int}(B(c,1)) \setminus \{c\}$, it follows that the degree of c is at most $\sum_{t=1}^{k} |N_t \setminus \operatorname{int}(B(c,1))| + k - 1 \leq (\vartheta(\mathcal{N})-1)k + k - 1 = \vartheta(\mathcal{N})k - 1$. $\square$

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