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PLANAR SEPARATORS*

NOGA ALON†, PAUL SEYMOUR‡, AND ROBIN THOMAS§

Abstract. The authors give a short proof of a theorem of Lipton and Tarjan, that, for every planar graph with $n > 0$ vertices, there is a partition (A, B, C) of its vertex set such that $|A|, |B| < \frac{2}{3}n$, $|C| \le 2(2n)^{1/2}$, and no vertex in A is adjacent to any vertex in B. Secondly, they apply the same technique more carefully to deduce that, in fact, such a partition (A, B, C) exists with $|A|, |B| < \frac{2}{3}n$, and $|C| \le \frac{3}{2}(2n)^{1/2}$: this improves the best previously known result. An analogous result holds when the vertices or edges are weighted.

Key words. separators, k-shields, corner

1. The Lipton–Tarjan theorem. Our first objective is to give a short proof of the following theorem of Lipton and Tarjan [3] ($V(G)$ denotes the vertex set of the graph G):

(1.1). *Let G be a planar graph with $n > 0$ vertices. Then there is a partition (A, B, C) of $V(G)$ such that $|A|, |B| < \frac{2}{3}n$, $|C| \le 2\sqrt{2}\sqrt{n}$, and no vertex in A is adjacent to any in B.*

Proof. We may assume that G has no loops or multiple edges, that $n \ge 3$, and (by adding new edges to G) that G is drawn in the plane in such a way that every region is bounded by a circuit of three edges. (Circuits have no "repeated" vertices.) Let $k = \lfloor \sqrt{2n} \rfloor$. For any circuit C of G, we denote by $A(C)$ and $B(C)$ the sets of vertices drawn inside C and outside C, respectively; thus $(A(C), B(C), V(C))$ is a partition of $V(G)$, and no vertex in $A(C)$ is adjacent to any in $B(C)$. Choose a circuit C of G such that

(i) $|V(C)| \le 2k$,
(ii) $|B(C)| < \frac{2}{3}n$,
(iii) subject to (i) and (ii), $|A(C)| - |B(C)|$ is minimum.

(This is possible because the circuit bounding the infinite region satisfies (i) and (ii).)

We suppose, for a contradiction, that $|A(C)| \ge \frac{2}{3}n$. Let D be the subgraph of G drawn in the closed disc bounded by C. For $u, v \in V(C)$, let $c(u, v)$ (respectively, $d(u, v)$) be the number of edges in the shortest path of C (respectively, D) between u and v.

(1) $c(u, v) = d(u, v)$ *for all* $u, v \in V(C)$.

For certainly, $d(u, v) \le c(u, v)$, since C is a subgraph of D. If possible, choose a pair $u, v \in V(C)$ with $d(u, v)$ minimum such that $d(u, v) < c(u, v)$. Let P be a path of D between u and v, with $d(u, v)$ edges. Suppose that some internal vertex w of P belongs to $V(C)$. Then

$$d(u, w) + d(w, v) = d(u, v) < c(u, v) \le c(u, w) + c(w, v),$$

and so either $d(u, w) < c(u, w)$ or $d(w, v) < c(w, v)$; either case is contrary to the choice of u, v. Thus there is no such w. Let C, C_1, C_2 be the three circuits of $C \cup P$.

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† Department of Mathematics, Sackler Faculty of Exact Sciences, Tel Aviv University, Ramat Aviv, Tel Aviv 69978, Israel. The research of this author was performed under a consulting agreement with Bellcore.

‡ Bellcore, 445 South Street, Morristown, New Jersey 07962.

§ School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332. This author was supported by Center for Discrete Mathematics and Theoretical Computer Science (DIMACS), Rutgers University, New Brunswick, New Jersey 08903.

where $|A(C_1)| \geq |A(C_2)|$. Now $|B(C_1)| < \frac{2}{3}n$, since

$$n - |B(C_1)| = |A(C_1)| + |V(C_1)|$$
$$> \tfrac{1}{2}(|A(C_1)| + |A(C_2)| + |V(P)| - 2) = \tfrac{1}{2}|A(C)| \geq \tfrac{1}{3}n.$$

However, $|V(C_1)| \leq |V(C)|$, since $|E(P)| \leq c(u, v)$, and so C_1 satisfies (i) and (ii). By (iii), $B(C_1) = B(C)$, and, in particular, $c(u, v) \leq 1$, which is impossible since $d(u, v) < c(u, v)$. This proves (1).

Suppose that $|V(C)| < 2k$. Choose $e \in E(C)$ and let P be the two-edge path of D such that the union of P and e forms a circuit bounding a region inside of C. Let v be the middle vertex of P and let P' be the path $C \backslash e$. Now $P \neq P'$, since $A(C) \neq \varnothing$, and so $v \notin V(C)$ by (1). Hence $P \cup P'$ is a circuit satisfying (i) and (ii), contrary to (iii). This proves that $|V(C)| = 2k$.

Let the vertices of C be $v_0, v_1, \ldots, v_{2k-1}, v_{2k} = v_0$, in order. There are $k + 1$ vertex-disjoint paths of D between $\{v_0, v_1, \ldots, v_k\}$ and $\{v_k, v_{k+1}, \ldots, v_{2k}\}$; for otherwise, by a well-known form of Menger's theorem for planar triangulations, there is a path of D between v_0 and v_k with $\leq k$ vertices, contrary to (1).

Let these paths be $P_0, P_1, \ldots, P_k$, where P_i has ends v_i, v_{2k-i} $(0 \leq i \leq k)$. By (1),

$$|V(P_i)| \geq \min(2i + 1, 2(k - i) + 1),$$

and so

$$n = |V(G)| \geq \sum_{0 \leq i \leq k} \min(2i + 1, 2(k - i) + 1) \geq \tfrac{1}{2}(k + 1)^2.$$

Yet $k + 1 > \sqrt{2n}$ by the definition of k, a contradiction. Thus our assumption that $|A(C)| \geq \frac{2}{3}n$ was false, and so $|A(C)| < \frac{2}{3}n$ and $(A(C), B(C), V(C))$ is a partition satisfying the theorem. □

2. Shields. In the remainder of the paper, we use the same technique more carefully to improve (1.1) numerically. A *separator* in a graph G is a partition (A, B, C) of $V(G)$ such that $|A|, |B| \leq \frac{2}{3}|V(G)|$ and no vertex in A is adjacent to any vertex in B; its *order* is $|C|$. Therefore, it is implied by (1.1) that any planar graph with n vertices has a separator of order $\leq 8^{1/2}n^{1/2}$, and we might try to find the smallest constant λ such that every planar graph with n vertices has a separator of order $\leq \lambda n^{1/2}$. The Lipton–Tarjan result (1.1) asserts that $\lambda \leq 8^{1/2} \simeq 2.828$, and this was improved by Gazit [2], who showed that $\lambda \leq \frac{7}{3} \simeq 2.333$. We give a further improvement, showing that $\lambda \leq \frac{3}{2} \cdot 2^{1/2} \simeq 2.121$. Incidentally, the best lower bound known appears to be that of Djidjev [1], who showed that

$$\lambda \geq \tfrac{1}{3}\sqrt{4\pi\sqrt{3}} \simeq 1.555.$$

Actually, we prove a slight strengthening, below (and indeed, we prove an extension when the vertices or edges have weights).

(2.1). *Let G be a loopless graph with n vertices, drawn in a sphere Σ. Then there is a simple closed curve F in Σ, meeting the drawing only in vertices, such that $n_1 + \frac{1}{2}n_3$, $n_2 + \frac{1}{2}n_3 \leq 2n/3$, and $n_3 \leq \frac{3}{2}(2n)^{1/2}$, where F passes through n_3 vertices and the two open discs bounded by F contain n_1 and n_2 vertices, respectively.*

We are concerned with graphs drawn in a disc or sphere Σ and, to simplify notation, we usually do not distinguish between a vertex of the graph and the point of Σ used in the drawing to represent the vertex, or between an edge and the open line segment representing it. A subset of Σ homeomorphic to the closed interval $[0, 1]$ is called