Advanced Combinatorics

Midterm Test, March 14, 2007

1. What is Hajós's necessary and sufficient condition for a graph G to have chromatic number at least k?

2. Let G be the *Petersen graph*, that is a 10-vertex graph G_P on the vertex set $V(G_P) = \{a_1, a_2, \ldots, a_5, b_1, b_2, \ldots, b_5\}$ with edges set $E(G_P) = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_1, b_1b_3, b_3b_5, b_5b_2, b_2b_4, b_4b_1, a_1b_1, a_2b_2, \ldots, a_5b_5\}$.

(a) Is G_P perfect?
(b) What is the chromatic number $\chi(G_P)$ of G_P?
(c) Is it true that $K_4 \prec G_P$, that is, K_4 is a minor of G_P?
(d) Is K_4 a topological minor of G_P?
(e) Is G_P 4-connected?

3. Prove that every k-connected graph with at least $2k$ vertices has a cycle of length at least $2k$.

4. A graph is called *k-edge-connected* if it has at least two vertices, it is connected, and it remains connected after the removal of any of its edges.

(a) Is it true that every 2-connected graph is 2-edge-connected?
(b) Is it true that every 2-edge-connected graph is 2-connected?

5. Prove that the vertices of every k-uniform hypergraph with at most $(3/2)^{k-3}$ hyperedges (k-tuples) can be colored by three colors such that every hyperedge contains points of all three colors.

Bonus questions for extra credit: 6. Let the vertex set of an infinite graph graph be

$$\{v_{i,j} : i = 1, 2, 3, \ldots \text{ and } j = 1, 2, 3, \ldots\}.$$

For every i and j, connect $v_{i,j}$ to all vertices $v_{k,i+j}$, $k = 1, 2, 3, \ldots$. Prove that the chromatic number of this graph is not finite, that is, no matter how we color its vertices by finitely many colors, there is an edge connecting two vertices of the same color.

7. Prove that for any graph G of n vertices, the sum of the chromatic numbers of G and its complement $\overline{G}$ is at most $n + 1$.

Please explain all of your answers! Good luck! - J.P.