3. Three partial orderings of the vertices of a plane graph

Here we follow Schnyder's approach to draw a plane graph on a grid.

__Definition 3.1.__ A __barycentric representation__ of a graph G is an assignment of triples (v_1, v_2, v_3) of real numbers to the vertices $v \in V(G)$ such that

(1) $(v_1, v_2, v_3) = (u_1, u_2, u_3) \Rightarrow v = u$

(2) $v_1 + v_2 + v_3 = 1 \quad (\forall v \in V(G))$

(3) for any $uv \in E(G)$ and for any $w \in V(G)$ $(w \neq u, v)$, there is some i such that $w_i > u_i$ and $w_i > v_i$.

Clearly, any barycentric representation of G gives rise to a straight-line drawing of G in the plane $x + y + z = 1$ (in $\mathbb{R}^3$).

__Proposition 3.2.__ This straight-line drawing is crossing-free.

__Proof.__ Take two disjoint edges $uv, wz \in V(G)$. Applying condition (3) to all triples uvw, uvz, wzu, wzv, two of the corresponding i values coincide. Suppose w.l.o.g. $w_i > u_i, v_i$ and $z_i > u_i, v_i$. But then the two edges can be separated by a line. ■

In what follows, we consider only maximal plane graphs, that is, triangulations G. Suppose G has a barycentric representation. Then we can label the angles of the interior faces (triangles) of G with the labels $1, 2, 3$ as follows:

$$\measuredangle uvw \text{ gets label } i \iff v_i > u_i, w_i$$

<u>Homework</u>: Show that this is a normal labeling.

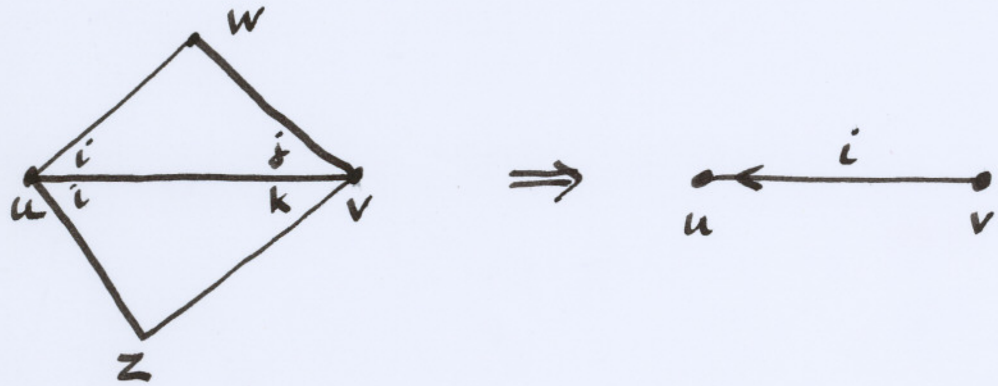
<u>Definition 3.3</u>. A labeling of the vertices of a triangulation G with $1, 2, 3$ is called <u>normal</u> if

(1) the angles of each interior triangle are labeled with $1, 2$, and 3, in counter-clockwise order; and

(2) at each interior vertex all three labels appear: an interval of 1's followed by an interval of 2's and an interval of 3's, in counterclockwise order.

We will see that every normal labeling can be obtained from a barycentric representation in the above manner.

But first we show that normal labelings of the interior <u>angles</u> is essentially equivalent to normal labelings of the interior <u>edges</u>. More precisely, consider an interior edge $uv \in E(G)$, at which two triangles uvw and vuz meet:



Either at u or at v the two triangles have angles with distinct labels. Suppose w.l.o.g. that the label of $\sphericalangle uvw$ is j, the label of $\sphericalangle uvz$ is $k \neq j$. Then both angles at u must receive the third label i. Now direct the edge uv from v to u, and label it with i. The directed edges of label i form a graph T_i $(i = 1, 2, 3)$. These graphs satisfy the conditions of the following definition.

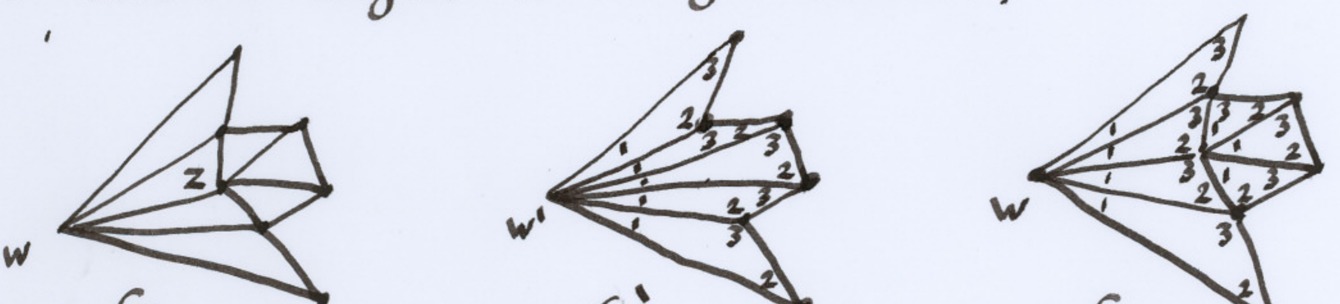
<u>Definition 3.4.</u> A <u>partition</u> of the interior edges of G into directed subgraphs T_1, T_2, T_3 is called <u>normal</u> if

(1) the out-degree of each interior vertex is 1 in T_i $(\forall i)$;

(2) the counterclockwise order of the edges around each interior vertex is: outgoing edges in T_1, incoming in T_3, outgoing in T_2, incoming in T_1, outgoing in T_3, incoming in T_2.

<u>Theorem 3.5.</u> Any triangulation admits a normal labeling of its interior angles.

<u>Proof</u>. Let w be a vertex of the exterior triangle in G. By Lemma 1.5, we can find an interior edge $wz \in E(G)$ that can be contracted without creating any parallel edges other than the images of the two triangles in G sitting on wz. We contract wz, and show by induction on the number of vertices the stronger statement that there is a labeling in which all interior angles at w get label 1.



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Illustrating the corresponding edge labeling ("normal partition" of edges):

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It also follows from this procedure that each T_i is a directed tree oriented towards a unique exterior vertex, called the <u>root</u> of T_i.

Let $\overline{T}_i$ denote the directed tree obtained from T_i by reversing the orientation of every edge. It remains true during the construction that $T_1 \cup \overline{T}_2 \cup \overline{T}_3$ has no directed cycle.

For each interior vertex $v \in V(G)$, follow the path $P_i(v) \subseteq T_i$ from v to the root of T_i $(i=1,2,3)$. These paths cannot meet, apart from the point v, e.g., by the last remark. So they divide the interior of G into three regions, $R_1(v)$, $R_2(v)$, and $R_3(v)$.

Notice that if another interior vertex $u \in R_i(v)$, then $R_i(u) \subsetneqq R_i(v)$. (Because of Definition 3.4 (2)!)

Define $\bar{v}_i$ $(i=1,2,3)$ as the number of triangles in the region $R_i(v)$. Clearly, we have $\bar{v}_1+\bar{v}_2+\bar{v}_3=2n-5$ for any interior vertex v (where n stands for the number of vertices of G). Extend this definition to the root w of T_1, by letting $\bar{w}_1 = 2n-5$, $\bar{w}_2=\bar{w}_3=0$, and similarly to the roots of T_2 and T_3. Now setting $v_i = \frac{\bar{v}_i}{2n-5}$ for every $v \in V(G)$, $i=1,2,3$, we obtain a barycentric representation of G.

<u>Homework</u>: Why?

Let $\underline{p}=(1,0)$, $\underline{q}=(0,1)$, $\underline{r}=(0,0)$ be regarded as vectors. Then $\bar{v}_1\underline{p}+\bar{v}_2\underline{q}+\bar{v}_3\underline{r} = (\bar{v}_1, \bar{v}_2)$ is a straight line drawing of G on a $(2n-5)\times(2n-5)$ grid.